Network Oblivious Transfer

Ranjit Kumaresan[*] Srinivasan Raghuraman[†] Adam Sealfon[‡]

Abstract

Motivated by the goal of improving the concrete efficiency of secure multiparty computation (MPC), we study the possibility of implementing an infrastructure for MPC. We propose an infrastructure based on oblivious transfer (OT), which would consist of OT channels between some pairs of parties in the network. We devise information-theoretically secure protocols that allow additional pairs of parties to establish secure OT correlations using the help of other parties in the network in the presence of a dishonest majority. Our main technical contribution is an upper bound that matches a lower bound of Harnik, Ishai, and Kushilevitz (Crypto 2007), who studied the number of OT channels necessary and sufficient for MPC. In particular, we characterize which n-party OT graphs G allow t-secure computation of OT correlations between all pairs of parties, showing that this is possible if and only if the complement of G does not contain the complete bipartite graph $K_{n-t,n-t}$ as a subgraph.

Keywords: Secure multiparty computation, oblivious transfer, dishonest majority, infrastructure.

1 Introduction

Protocols for secure multiparty computation [66, 31, 8, 16] allow a set of mutually distrusting parties to carry out a distributed computation without compromising the privacy of inputs or the correctness of the end result. As a research area, secure computation has witnessed several breakthroughs in the last decade [67, 47, 43, 59, 54, 53, 40, 57, 41, 52]. However, despite a wide array of potential game-changing applications, there is nearly no practical adoption of secure computation today (with the notable exceptions of [11, 12]). Computations wrapped in a secure computation protocol do not yet deliver results efficiently enough to be acceptable in many cloud-computing applications. For instance, state-of-the-art semihonest 2-party protocols incur a factor ≈ 100 slowdown even for simple computations.

In the absence of practical real-world protocols for secure computation which are secure in the presence of any number of dishonest parties, there is a need for relaxations that are meaningful and yet provide significant performance benefits. As an example, classic protocols for secure computation [8, 16, 63] (with subsequent improvements e.g., [19, 9, 4, 23, 21, 20]) offer vastly better efficiency at the cost of tolerating only a small constant fraction of adversaries. The resilience offered is certainly acceptable when the number of participating parties is large, e.g., the setting

[*] MIT. Email: ranjit@csail.mit.edu. Supported by the Qatar Foundation, MIT Translational Fellowship Program, ONR N00014-11-1-0486 and NSF CNS1413920.

[†] MIT. Email: srirag@csail.mit.edu. Supported by the Irwin Mark Jacobs and Joan Klein Jacobs Presidential Fellowship, NSF CNS1413920, DARPA W911NF-15-C-0236 and the Simons Foundation.

[‡] MIT. Email: asealfon@csail.mit.edu. Supported by a DOE CSGF fellowship, NSF CNS1413920, DARPA W911NF-15-C-0236 and the Simons Foundation.

of *large-scale* secure computation [13, 25, 68, 14]. Although large-scale secure computation is well-suited for several interesting applications (such as voting, census, surveys), we posit that typical settings involve computations over data supplied by a few end users. In such cases, the overhead associated with interaction among a large number of *helper parties* is likely to render these protocols more expensive than a standard secure computation protocol among the end users. If the number of helper parties is small, security against a small fraction of corrupt parties may be a very weak guarantee, since a handful of corrupt parties could render the protocol insecure.

An orthogonal approach for reducing the online cost of secure computation protocols is the use of *preprocessing* [3, 24, 10, 1]. This approach can dramatically reduce the cost of secure computation: for instance, given preprocessing [3], the ≈ 100 factor slowdown for simple computations no longer applies. Recent theoretical research has shown that many primitives can even be made *reusable* (e.g. [34]). Perhaps the most important drawback of this approach (other than the fact that the preprocessing phase is typically very expensive) is that the preprocessing is not *transferable*. Clearly, a pair of parties that want to perform a secure computation cannot benefit from this approach without performing the expensive preprocessing step. Moreover, this seems to hold even if each of the two parties have set up the preprocessing with multiple others. Typically, the cost of the preprocessing phase is quite high, presenting a barrier for the practical use of preprocessed protocols. This is especially true in settings where parties are unlikely to run many secure computations that would amortize the cost of preprocessing.

Motivated by the discussion above, we conclude that some directions that seem to offer efficiency benefits for secure computation are (1) highly resiliant protocols that use only a small number of helper parties, and (2) a preprocessing procedure that allows a notion of transferability between users. Taken together, these two ideas have the potential to provide an *infrastructure* for efficient secure computation. Some sets of parties might run a preprocessing phase among themselves. These parties can then act as helper parties and "transfer" their preprocessing to help users who want to run a secure computation protocol. We informally describe some desiderata for such an infrastructure:

- *Reusability/Amortization.* Setting up an infrastructure component could be expensive, but using it and maintaining it should be inexpensive relative to setting up a new component.
- *Transferability/Routing.* It should be possible to combine different components of the infrastructure to deliver benefits to the end users.
- *Robustness/Fault-tolerance.* Failure or unavailability of some components of the infrastructure should not nullify the usefulness of the infrastructure.

It is not hard to see that the above criteria are fulfilled for infrastructures that we use in daily life, for e.g., the infrastructure for online communication (e-mail, instant messaging, etc.) consisting of transatlantic undersea cables, routers, wireless access points, etc. What cryptographic primitives would be good candidates for a *secure computation infrastructure*? In this work, we explore the possibility of using *oblivious transfer* [62, 27] for this purpose.

1.1 Our Model: Network Oblivious Transfer

Oblivious transfer (OT) is a fundamental building block of secure computation [46, 45]. As discussed in [45], some of the benefits of basing secure computation on OT include:

- *Preprocessing.* OT enables precomputation in an offline stage before the inputs or the function to be computed are known. The subsequent online phase is extremely efficient [3].
- *Amortization.* The cost of computing OTs can be accelerated using efficient OT extension techniques [2, 43, 45, 59].
- *Security.* OTs can be realized under a wide variety of computational assumptions [60, 27, 62, 58, 18] or under physical assumptions.

In this work, we consider n parties connected by a synchronous network with secure point-to-point private communication channels between every pair of parties. In addition, some pairs of parties on the network have established *OT channels* between them providing them with the ability to perform arbitrarily many OT operations. We represent the OT channel network via an *OT graph* G. The vertices of G represent the n parties, and pairs of parties that have an established OT channel are connected by an edge in G. Since OT can be reversed unconditionally [64], we make no distinction between the sender and the receiver in an OT channel. This OT graph represents the infrastructure we begin with. The OT channels could either represent $\mathrm{poly}(\lambda)$ 1-out-of-2 OT correlations for a computational security parameter λ, or a physical channel (e.g., noisy channel) that realizes, say δ-Rabin OT [62].[1] We are interested in obtaining security against adaptive semihonest adversaries. We also discuss security against adaptive malicious adversaries under computational assumptions.

Two parties that are connected by an edge can use the corresponding existing OT channel to run a secure computation protocol between themselves. What about parties that are not connected by an edge? Clearly, they can establish an OT channel between themselves via an OT protocol [60, 18] or perhaps using a physical channel. The latter option, if possible, is likely to be expensive and the costs of setting up a physical channel may be infeasible unless the two parties are likely to execute many secure computation protocols. The former option is also expensive as it involves use of public-key cryptography which is somewhat necessary in the light of [42].[2] This motivates the question of whether additional parties can use an existing OT infrastructure to establish an OT channel between themselves unconditionally or relying only on the existence of symmetric-key cryptography. A positive result to this question would show that expensive cryptographic operations are not required to set up additional OT channels which could be used for efficient secure computation. In this work we construct OT protocols with information-theoretic security against a threshold adversary.

The generality of an OT infrastructure. Consider the following candidate for an infrastructure. Suppose there is a channel between a pair of parties that allows them to securely evaluate any function. Since OT is complete for secure computation, one can apply the results of [45, 46] to use the OT channel to implement a secure evaluation channel. In the other direction, one can use a secure evaluation channel to trivially implement OT channels. Consequently, such a channel is equivalent to an OT channel. The same argument extends to channels that implement any 2-party primitive that is complete for secure computation [55, 5]. Furthermore, the above argument also applies to the setting where a *set* of parties have a secure evaluation channel. Such a channel is

[1] Recall that λ 1-out-of-2 OT correlations can be extended to $\mathrm{poly}(\lambda)$ 1-out-of-2 OT correlations via OT extension using just symmetric-key cryptography (e.g. one-way functions [2] or correlation-robust hash functions [43]).

[2] As a rule of thumb, use of public-key cryptography is computationally around 4-6 orders of magnitude more expensive than using symmetric-key cryptography [7].

equivalent to an OT graph where parties in the set have pairwise OT channels with everyone in the set.

Assuming a full network of secure channels. Secure channels between two parties can be implemented either via non-interactive key exchange and hybrid encryption or via a physical assumption. We emphasize that the one-time setup cost of emulating a secure channel (e.g. via Diffie-Hellman key exchange) is much lower than the one-time setup cost of emulating an OT channel that allows unbounded OT calls via an OT protocol even using OT extension. Furthermore, our assumption of secure channels is identical to the setting of [46, 33, 45], who show that secure computation reduces to OT under information-theoretic reductions.

1.2 Related Work and Our Contributions

Related work. As mentioned previously, there is a large body of work on secure computation in the offline/online model (cf. [51, 50, 24, 10, 59] and references therein). These protocols exhibit an extremely fast online phase at the expense of a slow preprocessing phase (sometimes using MPC [51] or more typically, OT correlations [59] or a somewhat homomorphic encryption scheme [24]). To the best of our knowledge, the question of *transferability* of preprocessing has not been explicitly investigated in the literature with the notable exception of [36], which we will discuss in greater detail below. There is a large body of work on secure computation against a threshold adversary (e.g. [8, 16, 63, 31]). Popular regimes where secure computation against threshold adversaries have been investigated are for $t < n/3$, $t < n/2$, or $t = n - 1$. In this work we are interested in threshold adversaries for a dishonest majority, that is, adversaries which can corrupt t out of n parties for $n/2 \leq t < n$.[3] Such regimes were investigated in other contexts such as authenticated broadcast [29] and fairness in secure computation [6, 39, 44]. Infrastructures for *perfectly secure message transmission* (PSMT) were investigated in the seminal work of [26] (see also [28] and references therein). While the task of PSMT is similar to our question regarding OT channels, there are inherent differences. For example, our protocols can implement OT even between two parties that are isolated in the OT graph (i.e., not connected to any other party via an OT channel).[4] In PSMT, on the other hand, there is no hope of achieving secure communication with a node that is not connected by any secure channel.

Most relevant to our results is the work of Harnik, Ishai, and Kushilevitz [36]. The main question in their work is an investigation of the number of OT channels sufficient to implement a n-party secure computation protocol. In a nutshell, they show against an adaptive t-threshold adversary for $t = (1-\delta)n$, an explicit construction of an OT graph consisting of $(n+o(n))\binom{\lceil 1/\delta \rceil}{2}$ OT channels that suffices to implement secure computation among the n parties. They note further that against a static adversary, $\binom{\lceil s/\delta \rceil}{2}$ OT channels suffice, where s denotes a statistical security parameter. On the negative side, they show that a complete OT graph is necessary for secure computation when dealing with an adversary that can corrupt $t = n - 1$ parties. They derive this result by showing that in a 3-party OT graph with two OT channels, it is not possible to obtain OT correlations between the third pair of parties with security against two corruptions. Moreover they generalize their 3-party negative result to any OT graph whose complement contains the complete bipartite graph $K_{n-t,n-t}$ as a subgraph. In our paper we extend and generalize the results of [36], fully characterizing the networks for which it is possible to obtain OT correlations between a designated

[3] When $t < n/2$, there is no need to rely on an OT infrastructure [63].

[4] Recall that the model considered in this work, we assume a *full* network of secure private communication channels.

pair of parties. We now proceed to explain our contributions in more detail.

Our contributions. We introduce our main result:

Theorem (informal). Let $G = (V, E)$ be an OT graph on n parties $P_1, \dots P_n$, so that any pair of parties P_i, P_j which are connected by an edge may make an unbounded number of calls to an OT oracle. Let $\mathbb{A}$ be the class of semihonest t-threshold adversaries which may adaptively corrupt at most t parties.[5] Then two parties A and B in $\{P_1, \dots, P_n\}$ can information-theoretically emulate an OT oracle while being secure against all adversaries $\mathcal{A} \in \mathbb{A}$ if and only if

1. (honest majority) it holds that $t < n/2$; or
2. (trivial) A and B are connected by an edge in G; or
3. (partition) there exists no partition V_1, V_2, V_3 of G such that all of the following conditions are satisfied: (a) $|V_1| = |V_2| = n - t$ and $|V_3| = 2t - n$; (b) $A \in V_1$ and $B \in V_2$; and (c) for every $A' \in V_1$ and $B' \in V_2$ it holds that $(A', B') \notin E$.

Our main theorem gives a complete characterization of networks for which a pair of parties can utilize the OT network infrastructure to execute a secure computation protocol. The first two conditions in our theorem are straightforward: (1) if $t < n/2$, then we are in the honest majority regime, and thus it is possible to implement secure computation (or emulate an OT oracle) using the honest majority information-theoretically secure protocols of [63]; (2) clearly if A and B are connected by an OT edge then by definition they can emulate an OT oracle.

Condition (3) applies when $t \geq n/2$ and when A and B do not have an OT edge between them. This condition is effectively the converse of the impossibility result of [36], which states that any n-party OT graph whose complement contains $K_{n-t,n-t}$ as a subgraph cannot allow a n-party secure computation that tolerates t semihonest corruptions. Condition (3) implies that any n-party OT graph whose complement does not contain $K_{n-t,n-t}$ as a subgraph can run n-party secure computations tolerating t semihonest corruptions.

Applying our main theorem. We first compare our positive results to those of [36]. They investigate how to construct an OT graph with the minimum number of edges allowing n parties to execute a secure computation protocol. They show a construction for a graph with $(n+o(n))\binom{\lceil 1/\delta \rceil}{2}$ edges which they prove is sufficient for resilience against an adversary that corrupts $(1-\delta)n$ parties. Our result provides a complete, simple characterization of which OT graphs on n vertices are sufficient to run a t-secure protocol generating OT correlations between all pairs of vertices for any $t \geq n/2$, which is sufficient to obtain a protocol for secure computation among the n parties [46, 45]. Our main theorem also implies that determining the minimum number of OT edges needed to execute a secure computation protocol for general $n, t \geq n/2$ is equivalent to an open problem in graph theory posed by Zarankiewicz in 1951 [48].

Our results immediately imply that for some values of t, extremely simple sparse OT graphs suffice for achieving secure multiparty computation. For n even and $t = n/2$, we have that the t-claw graph (cf. Fig. 4(a)) has t edges and suffices to achieve t-secure multiparty computation. For n odd and $t = (n+1)/2$, the $(t+1)$-cycle has $t+1$ edges and suffices to achieve t-secure multiparty computation. We show in Appendix A that these examples are the sparsest possible graphs which can achieve $\lfloor (n+1)/2 \rfloor$-secure multiparty computation.

[5] Combining our work with results from [35, 32], we can also obtain computational security against malicious adversaries in both the nonadaptive and adaptive settings.

Next, our results are also well-suited to make use of an OT infrastructure for secure computation. Specifically, let G_I denote the OT graph consisting of existing OT edges between parties that are part of the infrastructure. Now suppose a pair of parties A, B not connected by an OT edge wish to execute a secure computation protocol. Then they can find a subgraph G of G_I with $A, B \in G$ and $|G| = n$ such that they agree that at most t out of the n parties can be corrupt and the partition condition in our main theorem holds for G. Since it is possible to handle a dishonest majority, parties do not have to settle for a lower threshold and can enjoy increased confidence in the security of their protocol by making use of the infrastructure. Surprisingly, it turns out the OT subgraph G need not even contain t OT edges to offer resilience against t corruptions (cf. Fig. 2(c) with $n = 4, t = 2$).

A pair of parties may use the OT correlations generated as the base OTs for an OT extension protocol and inexpensively generate many OT correlations that can be saved for future use or to add to the OT infrastructure. In any case, it should be clear that our protocols readily allow load-balancing across the OT infrastructure and are also abort-tolerant in the sense that if some subgraph G ends up not delivering the output, then one can readily use a different subgraph G'. Thus we believe that our results can be used to build a *scalable* infrastructure for secure computation that allows (1) amortization, (2) routing, and (3) is robust.

An important caveat regarding efficiency. In the special cases $t = n/2 + \mathcal{O}(1)$ and $t = n - \mathcal{O}(1)$, determing whether a graph satisfies the partition condition requires at most $\text{poly}(n)$ time. However, in general the problem is coNP-complete, since it can be restated in the graph complement as subgraph isomorphism of a complete bipartite graph [30]. Our protocols are efficient in n only for $t = n/2 + \mathcal{O}(1)$ and $t = n - \mathcal{O}(1)$.[6] In particular, our protocol is quite efficient for small values of n, a setting in which computing OT correlations in the presence of a dishonest majority may be especially useful in practice.

Organization After discussing preliminaries in Section 2, we give an overview of some of our techniques in Section 3, where we show solutions for the simple case when $n = 4$ and $t = 2$. Following this we briefly sketch the lower bound in Section 4 and describe the building blocks required for our upper bounds in Section 5. The rest of the paper is devoted to proving the upper bound first for the specific cases of $t = n/2$ (Section 6) and $t = n - 2$ (Section 7). We then use each of these protocols in two different solutions for the general case of $t \geq n/2$ in Section 8 which are efficient for different values of the corruption threshold t.

2 Preliminaries

2.1 Notation and definitions

Let $\mathcal{X}, \mathcal{Y}$ be two probability distributions over some set S. Their *statistical distance* is

$$\mathbf{SD}\left(\mathcal{X}, \mathcal{Y}\right) \stackrel{\text{def}}{=} \max_{T \subseteq S}\{\Pr\left[\mathcal{X} \in T\right] - \Pr\left[\mathcal{Y} \in T\right]\}$$

We say that $\mathcal{X}$ and $\mathcal{Y}$ are ϵ-close if $\mathbf{SD}\left(\mathcal{X}, \mathcal{Y}\right) \leq \epsilon$ and this is denoted by $\mathcal{X} \approx_\epsilon \mathcal{Y}$. We say that $\mathcal{X}$ and $\mathcal{Y}$ are identical if $\mathbf{SD}\left(\mathcal{X}, \mathcal{Y}\right) = 0$ and this is denoted by $\mathcal{X} \equiv \mathcal{Y}$.

[6] For $t = n/2 + \mathcal{O}(1)$, we achieve efficiency using computationally-secure OT extension (e.g. [2, 43]). Our protocol with information-theoretic security is quasipolynomial-time for $t = n/2 + \mathcal{O}(1)$. We do, however, achieve information-theoretic security in polynomial time for $t = n - \mathcal{O}(1)$.

All graphs addressed in this work are undirected. We denote a graph as $G = (V, E)$ where V is a set of vertices and E is a set of edges. We denote an edge e as $e = \{v_1, v_2\}$, where $v_1, v_2 \in V$.

For $n \in \mathbb{N}$, let K_n denote the complete graph on n vertices. Let Λ_a^s denote the graph $G = (V, E)$ on $2a + s$ vertices with $V = V_A \dot{\bigcup} V_S \dot{\bigcup} V_B$, where $|V_A| = |V_B| = a$ and $|V_S| = s$, and

$$E = \{\{v_1, v_2\} : v_1 \notin V_A \vee v_2 \notin V_B\}$$

We will sometimes consider subgraphs of Λ_a^s which preserve labels of vertices. In this case we will always label the vertices so that vertex $A \in V_A$ and vertex $B \in V_B$.

For two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set V, we say that G_1 and G_2 are $(v_1, \ldots, v_\ell)$-*isomorphic*, denoted by $G_1 \simeq_{v_1,\ldots,v_\ell} G_2$, if the two graphs are isomorphic to one another while fixing the labelings of vertices $v_1, \ldots, v_\ell \in V$, that is, there exists an isomorphism σ such that $\sigma(v_i) = v_i$ for all $i \in [\ell]$.

Similarly, given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \subseteq V_2$ and $v_1, \ldots, v_\ell \in V_1$, we say that G_1 is a $(v_1, \ldots, v_\ell)$-*subgraph* of G_2, denoted $G_1 \subseteq_{v_1,\ldots,v_\ell} G_2$, if G_1 is $(v_1, \ldots, v_\ell)$-isomorphic to some subgraph of G_2.

In particular, in the special case that graph $G = (V, E)$ contains vertices $A, B \in V$, we say that G is an (A, B)-*subgraph* of Λ_a^s (or that $G \subseteq_{A,B} \Lambda_a^s$) if there is an isomorphism σ between G and a subgraph of Λ_a^s such that A is mapped into set V_A and B is mapped into set V_B (that is, $\sigma(A) \in V_A$ and $\sigma(B) \in V_B$).

Call an n-vertex graph $G = (V, E)$ k-*unsplittable* for $k \leq n/2$ if any two disjoint sets of k vertices have some edge between them. That is, G is k-unsplittable if for all partitions of the vertices V into three disjoint sets V_1, V_2, V_3 of sizes $|V_1| = |V_2| = k$ and $|V_3| = n - 2k$, there exists some edge $(u, v) \in E$ with $u \in V_1, v \in V_2$. It is immediate from this definition that G is k-unsplittable if and only if $G \not\subseteq \Lambda_k^{n-2k}$.

Similarly, call G (k, A, B)-*unsplittable* for $k \leq n/2$ and $A, B \in V$ if any two disjoint sets of k vertices containing A and B, respectively, have some edge between them. That is, G is (k, A, B)-unsplittable if for all partitions of the vertices of V into three disjoint sets V_1, V_2, V_3 of sizes $|V_1| = |V_2| = k$ and $|V_3| = n - 2k$ such that $A \in V_1$ and $B \in V_2$, there exists some edge $(u, v) \in E$ with $u \in V_1, v \in V_2$. From this definition we have immediately that G is (k, A, B)-unsplittable if and only if $G \not\subseteq_{A,B} \Lambda_k^{n-2k}$.

2.2 Secure Computation

Consider the scenario of n parties $P_1, \ldots, P_n$ with private inputs $x_1, \ldots, x_n \in \mathcal{D}$ computing a function $f : \mathcal{D}^n \to \mathcal{D}^n$. Let Π be a protocol computing f. We consider security against adaptive t-threshold adversaries, that is, adversaries that adaptively corrupt a set of at most t parties, where $0 \leq t < n$.[7] We assume the adversary to be semihonest (i.e. honest-but-curious). That is, the corrupted parties follow the prescribed protocol, but the adversary may try to infer additional information about the inputs of the honest parties. As noted in [36], in the computational setting, using zero-knowledge proofs, it is possible to generically compile a protocol which is secure against semihonest adversaries into another protocol which is secure against adaptive malicious adversaries [32].[8] This justifies our focus on the semihonest setting here.

[7] Note that when $t = n$, there is nothing to prove.

[8] We note that in the computational setting, it is also possible to transform, in a *black-box* way, a protocol which is secure against semihonest adversaries into another protocol which is secure against static malicious adversaries [35].

For a PPT adversary $\mathcal{A}$, let random variable $\mathrm{REAL}_{\Pi,\mathcal{A}}^{x_1,\ldots,x_n}$ consist of the views of the corrupted parties when the protocol Π is run on parties $P_1,\ldots,P_n$ with inputs $x_1,\ldots,x_n$ respectively. In the ideal world, the honest parties are replaced with a simulator $\mathcal{S}$ that does not receive input values and knows only the output value of each corrupted party in an honest execution of the protocol. We define the random variable $\mathrm{IDEAL}_{\Pi,\mathcal{A},\mathcal{S}}^{x_1,\ldots,x_n}$ as the output of the adversary $\mathcal{A}$ in the ideal game with the simulator when the inputs to parties $P_1,\ldots,P_n$ are $x_1,\ldots,x_n$, respectively.

Definition 1. *A protocol Π is said to t-securely compute the function f if*

- *For all $x_1,\ldots,x_n \in \mathcal{D}^n$, party P_i receives y_i, where $(y_1,\ldots,y_n) = f(x_1,\ldots,x_n)$, at the end of the protocol.*
- *For all adaptive semihonest PPT t-threshold adversaries $\mathcal{A}$, there exists a PPT simulator $\mathcal{S}$ such that for all $x_1,\ldots,x_n \in \mathcal{D}^n$*

$$\left\{\mathrm{REAL}_{\Pi,\mathcal{A}}^{x_1,\ldots,x_n}\right\} \equiv \left\{\mathrm{IDEAL}_{\Pi,\mathcal{A},\mathcal{S}}^{x_1,\ldots,x_n}\right\}$$

This definition is for secure computation with perfect information-theoretic security and a nonadaptive adversary. By [15], in the semihonest setting with information-theoretic security, any protocol which is nonadaptively secure is also adaptively secure. Consequently, satisfying this definition suffices to achieve adaptive security.

In the discussion below, we will sometimes relax security to statistical or computational definitions. A protocol is statistically t-secure if the random variables $\mathrm{REAL}_{\Pi,\mathcal{A}}^{x_1,\ldots,x_n}$ and $\mathrm{IDEAL}_{\Pi,\mathcal{A},\mathcal{S}}^{x_1,\ldots,x_n}$ are statistically close, and computationally t-secure if they are computationally indistinguishable.

2.3 Oblivious Transfer

In this work OT refers to 1-out-of-2 oblivious transfer defined as follows.

Definition 2. *We define 1-out-of-2 oblivious transfer f_{OT} for a sender $A = P_1$ with inputs $x_0, x_1 \in \{0,1\}^m$, a receiver $B = P_2$ with input $b \in \{0,1\}$ and $n-2$ parties $P_3,\ldots,P_n$ with input $\perp$ as*

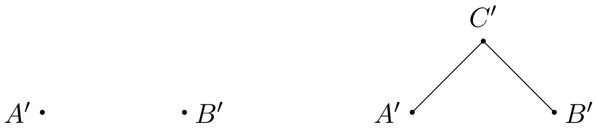
$$f_{\mathrm{OT}}((x_0,x_1),b,\perp,\ldots,\perp) = (\perp,x_b,\perp,\ldots,\perp)$$

Note that while OT is typically defined as a 2-party functionality, the definition above adapts it our setting and formulates OT as an n-party functionality where only two parties supply non-$\perp$ inputs.

Definition 3. *Let G be a network consisting of n parties $A = P_1, B = P_2, P_3,\ldots,P_n$. Then a t-secure OT protocol $\Pi_{A\to B}^{G,t}$ is a protocol that t-securely computes the function f_{OT} on the inputs of the parties with A as the sender and B as the receiever.*

We note that OT is symmetric, in the following sense.

Lemma 1. *[64] If there exists a t-secure OT protocol $\Pi_{A\to B}^{G,t}$ for an n-party network G with n parties $A = P_1, B = P_2, P_3,\ldots,P_n$ with A as the sender and B as the receiever, then there exists a t-secure OT protocol $\widehat{\Pi}_{B\to A}^{G,t}$ for the same n parties with B as the sender and A as the receiever.*



(a) G_{CK}

(b) G_{HIK}

Figure 1: Known impossibility results. Securely computing f_{OT} between A' and B' is impossible for $t = 1$ in G_{CK} and is impossible for $t = 2$ in G_{HIK}.

We represent parties as nodes of a graph G where an edge $\{A, B\}$ indicates that parties A and B may run a 1-secure OT protocol with A as the sender and B as the receiver. By Lemma 1, the roles of the sender and receiver may be reversed, so it makes sense to define G as an undirected graph.

We note the following result regarding the completeness of OT for achieving arbitrary secure multiparty computation.

Lemma 2. *[46, 33, 45] Consider the complete network $G \simeq K_n$ on n vertices. Then, for any function $f : \mathcal{D}^n \to \mathcal{R}^n$, there exists a protocol Π which $(n-1)$-securely computes f, where party i receives the ith input $x_i \in \mathcal{D}$ and produces the ith output $(f(x))_i \in \mathcal{R}$.*

3 Warm-ups

Let $G = (V, E)$ be an n-vertex graph representing a network with n parties, where an edge $\{P_i, P_j\} \in E$ indicates that parties P_i and P_j may run a 1-secure 2-party OT protocol with P_i as the sender and P_j as the receiver. Let $t < n$ be an upper bound on the number of corruptions made by the adversary. The central question considered in this work is the following. For which graphs G and which pairs of parties $A, B \in V$ does there exist a t-secure OT protocol with A as the sender and B as the receiver?

We begin by discussing some simple special cases of small networks. These will provide useful intuition for our main results. For $t < n/2$, it is possible to obtain a t-secure OT protocol for any n-vertex graph $G = (V, E)$ between any $A, B \in V$, since we can perform secure multiparty computation without any pre-existing OT channels if there is an honest majority [63]. It remains to consider the setting where $t \geq n/2$.

A few small cases have been resolved in prior work. For $n = 2$, $t = 1$, a 1-secure OT protocol (with perfect security) between the vertices of the two-vertex graph G does not exist unless the parties were already connected by an OT channel [17, 49]. This result is illustrated in Figure 1(a).

For $n = 3$, $t = 2$, it is known that we can obtain a 2-secure OT protocol between a pair of vertices A, B only if those vertices are already connected by an OT channel, even if there are OT channels from both A and B to the third vertex C as depicted in Figure 1(b). More generally, for any $n \geq 2$ and $t = n - 1$, there exists a t-secure OT protocol with sender A and receiver B only if those vertices are already connected by an OT channel, even if all other $\binom{n}{2} - 1$ pairs of vertices are connected by OT channels [36]. This also resolves the question for $n = 4, t = 3$.

The remainder of this section is devoted to an exploration of the setting $n = 4, t = 2$. This is the smallest case not resolved by prior techniques, and will illustrate many of the tools used in

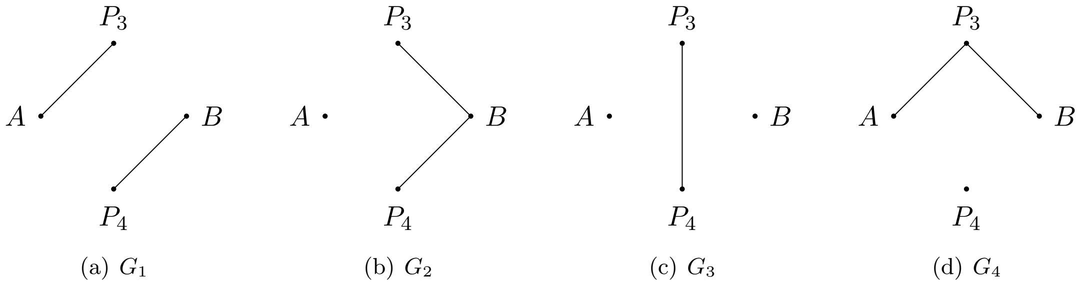


Figure 2: Cases for $n = 4$ parties with $t = 2$ corruptions.

subsequent sections to obtain our general protocols. The key cases for $n = 4, t = 2$ are shown in Figure 2. As discussed below, these cases are sufficient to completely resolve the four-party setting.

3.1 Case 1 : Figure 2(a)

We first show that if $G \simeq_{A,B} G_1$ then there does not exist a 2-secure OT protocol for G with A as the sender and B as the receiver.[9] This is a consequence of the impossibility result of [17, 49]. An outline of the argument is as follows.

Consider components $\mathcal{C}_1 = \{A, P_3\}$ and $\mathcal{C}_2 = \{B, P_4\}$ of G, and let Π be a 2-secure protocol computing f_{OT} in G with A as the sender and B as the receiver. Then we can use Π to construct a 1-secure protocol Π' for the 2-party network G_{CK} in Figure 1(a) with A' as the sender and B' as the receiver. In protocol Π', party A' runs Π for both parties of component $\mathcal{C}_1$ of G, and B' runs Π for both parties of component $\mathcal{C}_2$. OT channel invocations can be handled locally, since all OT channels in G are between parties in the same component. Since protocol Π is 2-secure, in particular it is secure against corruptions of parties in $\mathcal{C}_1$ or the parties in $\mathcal{C}_2$. Consequently Π' is a 1-secure OT protocol for a network $G' \simeq_{A',B'} G_{\text{CK}}$ with A' as the sender and B' as the receiver. However, from [17, 49], we know that no such protocol exists with perfect security. Consequently there is no 2-secure protocol Π for a network $G \simeq_{A,B} G_1$.

Note that this impossibility holds not only for $G \simeq_{A,B} G_1$ but for any (A, B)-subgraph of G_1. In particular, if $G = (V, E)$ is a four-vertex graph a single edge that is incident to vertex A or vertex B, then G cannot have a 2-secure protocol computing f_{OT} between A and B except in the trivial case when there is already an edge $\{A, B\} \in E$. This technique of reducing to the known impossiblity results of [17, 49, 36] to obtain lower bounds is described formally in Section 4.

3.2 Case 2 : Figure 2(b)

In this example we obtain a positive result, showing that there exists a 2-secure OT protocol with A as the sender and B as the receiver. Since B has degree 2 in G_2, we have that either B or one of its neighbors must be honest, and so one of the two OT channels must contain an honest party. This suggests the idea of using secret-sharing to ensure security against 2 corruptions.

Consider the following OT protocol where sender A has inputs $x_0, x_1 \in \{0,1\}^m$ and receiver B has input $b \in \{0,1\}$. A computes 2-out-of-2 shares (x_0^1, x_0^2) and (x_1^1, x_1^2) of its inputs x_0, x_1,

[9] Recall that $H \simeq_{A,B} H'$ for two graphs H, H' if there exists an isomorphism between H and H' preserving the labels of vertices A and B.

respectively. A then sends shares x_0^1 and x_1^1 to party P_3 and x_0^2 and x_1^2 to party P_4. Parties P_3 and B invoke their secure OT channel with inputs (x_0^1, x_1^1) and b, and parties P_4 and B invoke their secure OT channel with inputs (x_0^2, x_1^2) and b respectively. B uses the obtained shares x_b^1, x_b^2 to reconstruct x_b.

We informally argue the 2-security of this protocol assuming that exactly one of A and B is corrupt.[10] Consider the case where A is corrupt and B is honest. The input of B is only used over secure OT channels, so by the 1-security of the OT channels with P_3 and P_4, the corrupt parties can learn nothing about B's input bit b. Now consider the case where B is corrupt and A is honest. Either P_3 or P_4 must be honest. If P_3 is honest then the security of OT channel $\{P_3, B\}$ implies that B learns nothing about share x_{1-b}^1, so the security of the secret sharing scheme implies that the corrupt parties do not use x_{1-b}. By symmetry, the same argument applies if P_4 is honest. This completes the argument.

Note that by Lemma 1, we can also obtain a 2-secure OT protocol from A to B whenever A has degree 2 in OT network. Furthermore, we can extend this idea to construct a t-secure OT protocol whenever either the sender or the receiver has degree at least t. We call this protocol the t-claw protocol and describe it in detail in Section 5.1.

3.3 Case 3 : Figure 2(c)

Somewhat surprisingly, we can also show a positive result for graphs $G \simeq_{A,B} G_3$ even though the OT network has no edges involving either the sender A or the receiver B. The protocol is as follows. Since parties P_3 and P_4 have an OT channel between them, by Lemma 2, they can perform 1-secure MPC between them. P_3 and P_4 use MPC to compute 2-out-of-2 shares of OT correlations with uniformly random inputs and send corresponding shares to A and B, who can then reconstruct the correlations. More concretely, the MPC protocol computes 2-out-of-2 shares (r_0^1, r_0^2), (r_1^1, r_1^2) of two randomly sampled m-bit strings r_0, r_1, 2-out-of-2 shares (c^1, c^2) of a random bit $c \in \{0,1\}$, and independent 2-out-of-2 shares (s^1, s^2) of the string r_c. Party P_3 receives the first share of each secret, and party P_4 receives the second share. Party P_3 then sends shares r_0^1, r_1^1 to A and s^1, c^1 to B, while P_4 sends shares r_0^2, r_1^2 to A and s^2, c^2 to B. A can then reconstruct r_0 and r_1, and B can reconstruct c and r_c. Parties A and B have now established a random OT correlation, which they can use to perform OT with their original inputs using OT correction [3].[11]

We now informally argue the 2-security of this protocol. If A and B are both honest, then the corrupt parties receive no information about their inputs, while if A and B are both corrupt then there is nothing to prove. Consequently we can assume that exactly one of A and B is corrupt and that either P_3 or P_4 is honest. If A is corrupt and P_3 or P_4 is honest, then the adversary learns nothing about c and r_c, since it only sees one of the two shares of each. The OT correction phase uses these strings as one-time pads for inputs which are unknown to the adversary, and consequently are information-theoretically hidden from the adversary. Consequently A learns nothing about B. The case where B is corrupt and P_3 or P_4 is honest follows by the same argument.

[10] An additional step is needed to address the case in which P_3 and P_4 are corrupt and A and B are both honest. Then P_3 and P_4 can learn x_0 and x_1, the inputs of A, in the protocol just described. This can be handled with the technique of OT correction, using a one-time pad and the secure point-to-point channel between A and B. Equivalently, we could run the protocol on random inputs, and then use method of [3] to obtain 1-out-of-2 OT from random OT. If A and B are both corrupt then there is nothing to prove.

[11] This OT correction step can be performed as follows. Party B sends $b' = b \oplus c$ to A. A responds with $y_0 = x_0 \oplus r_{b'}$ and $y_1 = x_1 \oplus r_{1-b'}$. Finally, B computes $y_b \oplus r_c = x_b$.

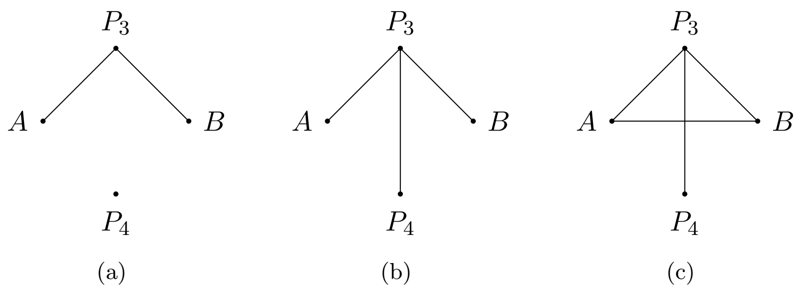


Figure 3: Illustrating the cascading protocol for Case 4 : Figure 2(d); (a) → (b) → (c)

This construction can be extended to obtain a t-secure OT protocol whenever the OT graph contains a t-clique consisting of t parties which are not the OT sender or receiver. We call this protocol the t-clique protocol and describe it in detail in Section 5.2.

3.4 Case 4 : Figure 2(d)

We also obtain a positive result for graphs $G \simeq_{A,B} G_4$. We introduce here a technique we call cascading. The idea is as follows. Using the protocol described in Section 3.2 for network G_2 of Figure 2(b), we have 2-secure OT protocol with P_3 as the sender and P_4 as the receiver. This effectively gives us an OT channel between P_3 and P_4. Applying the protocol from Section 3.3 on the augmented network, we obtain a 2-secure OT protocol with A as the sender and B as the receiver. We describe this pictorially in Figure 3.

The 2-security of the protocol follows from the 2-security of the underlying protocols of Sections 3.2 and 3.3. The technique of cascading for combining t-secure protocols is described in detail in Section 5.3.

3.5 Cases 1–4 are exhaustive

Note that a t-secure OT protocol with sender A and receiver B in an OT network G trivially yields a t-secure protocol for any network G' such that $G \subseteq_{A,B} G'$. From cases 1 and 3, we can securely compute f_{OT} in a network G containing at most a single edge if and only if the edge is $\{A, B\}$ or $\{P_3, P_4\}$. From cases 1, 2, and 4, we can compute f_{OT} in a network G containing two or more edges including neither of $\{A, B\}$ or $\{P_3, P_4\}$ if and only if there is some vertex with degree at least 2 in the OT graph. This completes the characterization of 4-party networks with 2 corruptions.

4 Lower Bound

We now describe a family of impossibility results using a generic reduction to the impossiblity result in [36], which we restate in our language below.

Lemma 3. *[36] Consider any three party network G with $G \simeq_{A',B'} G_{\text{HIK}}$, the graph in Figure 1(b). Then any 2-secure OT protocol with A' as the sender and B' as the receiver can be used (as a black box) to obtain a 1-secure OT protocol for a network G' with $G' \simeq_{A',B'} G_{\text{Kus}}$, the graph in Figure 1(a), with A' as the sender and B' as the receiver.*

The theorem below describes an impossibility result over a family of networks. We note that this result was observed in [36]; we restate it our language and provide a formal proof.

Theorem 4. *Let $n \geq 2$ and $n/2 \leq t < n$, and let G be an n party network such that $G \subseteq \Lambda_{n-t}^{2t-n}$, with $P_1 \in V_A$ and $P_2 \in V_B$. Any t-secure OT protocol for G with P_1 as the sender and P_2 as the receiver can be used (as a black box) to obtain a 1-secure OT protocol for a network G' with $G' \simeq_{A,B} G_{\text{CK}}$ with A' as the sender and B' as the receiver.*

Proof. Let G be an n party network with $G = (V, E)$ such that $G \subseteq \Lambda_{n-t}^{2t-n}$. Then, we may write $V = V_A \dot{\bigcup} V_S \dot{\bigcup} V_B$, where $|V_A| = |V_B| = n - t$ and $|V_S| = 2t - n$, with $P_1 \in V_A$ and $P_2 \in V_B$ and $E(V_A, V_B) = \emptyset$, where $E(V_A, V_B)$ represents the set of edges with one endpoint in V_A and the other in V_B.

Let Π be a t-secure OT protocol for G with P_1 as the sender and P_2 as the receiver. If $t > n/2$, then we can use Π to construct a 2-secure OT protocol Π' for any three party network G' with $G' \simeq_{A',B'} G_{\text{HIK}}$ with A' as the sender and B' as the receiver below. Combining this with Lemma 3, the conclusion follows. We describe the construction of Π' below. If $t = n/2$, then we can use Π to construct a 1-secure OT protocol Π'' for any two party network G'' with $G'' \simeq_{A'',B''} G_{\text{CK}}$ with A'' as the sender and B'' as the receiver. The construction of Π'' is exactly the same as that of Π' and hence we omit its description here.

In protocol Π', party A' simulates the parties of component V_A, party C' simulates the parties of component V_S, and party B' simulates the parties of component V_B. Executions of 1-secure OT protocols between parties of the same component are handled locally and executions of 1-secure OT protocols between parties in different components are handled as follows:

- If the parties are in components V_A and V_S, then executions of 1-secure OT protocols between the parties are carried out using the OT edge $\{A', C'\}$ in the network G'.

- If the parties are in components V_B and V_S, then executions of 1-secure OT protocols between the parties are carried out using the OT edge $\{B', C'\}$ in the network G'.

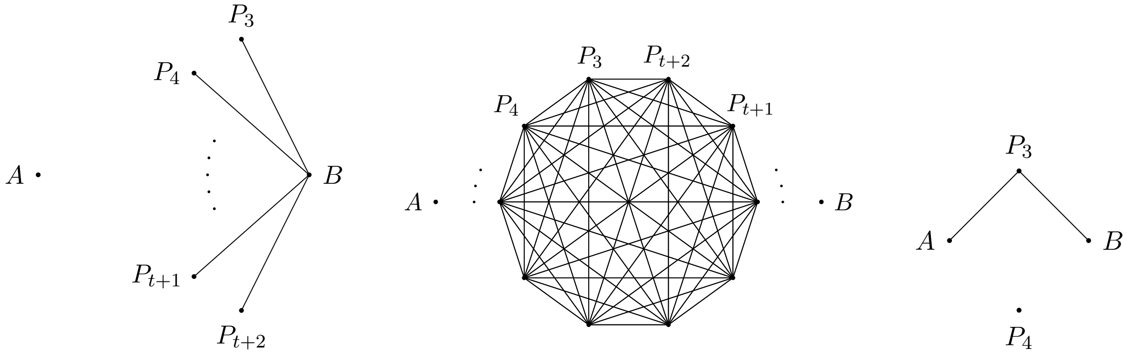
Since $G \subseteq \Lambda_{n-t}^{2t-n}$, there are no executions of 1-secure OT protocols between parties in components V_A and V_B in protocol Π.

Correctness of Π' is obvious. We now prove 2-security of Π'. Intuitively, since Π is t-secure, in particular, it is secure against corruptions of parties $V_A \cup V_S$ or the parties $V_B \cup V_S$. Consequently protocol Π' is secure against corruptions $\{A', C'\}$ or $\{B', C'\}$ and hence Π' is a 2-secure protocol.

Formally, we prove this as follows. Since Π is t-secure, there exists a simulator $\mathcal{S}$ such that for every PPT t-threshold adversary $\mathcal{A}$, $\left\{\text{REAL}_{\Pi,\mathcal{A}}^{(x_0,x_1),b,\perp,\ldots,\perp}\right\} \equiv \left\{\text{IDEAL}_{\Pi,\mathcal{A},\mathcal{S}}^{(x_0,x_1),b,\perp,\ldots,\perp}\right\}$, where $x_0, x_1 \in \{0,1\}^m$ and $b \in \{0,1\}$. The simulator $\mathcal{S}'$ for the protocol Π' behaves exactly the same as $\mathcal{S}$ while simulating the parties of component V_A as A', those of of component V_B as B' and those of of component V_S as C'. It is easy to see that for any 2-threshold adversary $\mathcal{A}'$, namely, one which corrupts $\{A', C'\}$ or the one which corrupts $\{B', C'\}$, $\left\{\text{REAL}_{\Pi',\mathcal{A}'}^{(x_0,x_1),b,\perp}\right\} \equiv \left\{\text{IDEAL}_{\Pi',\mathcal{A}',\mathcal{S}'}^{(x_0,x_1),b,\perp}\right\}$ since Π is secure against corruptions of parties $V_A \cup V_S$ or the parties $V_B \cup V_S$. □

5 Building Blocks

In this section, we describe a few key protocols and techniques that we use in the subsequent sections to prove our main theorem.



(a) G^t_{claw} (b) G^t_{clique} (c) $G^2_{\text{2-path}}$

Figure 4: Building block networks. (a) t-claw graph (b) t-clique graph (c) 2-path graph

Protocol 1: t-claw Protocol

Preliminaries: Let $A, B, P_3, \ldots, P_{t+2}$ be the $t+2$ parties in a network $G \simeq_{A,B} G^t_{\text{claw}}$. A has inputs $x_0, x_1 \in \{0,1\}^m$ and B has input $b \in \{0,1\}$.

Protocol:

1. B chooses a random bit $c \in \{0,1\}$ and sends $b' = b \oplus c$ to A.
2. A chooses two random one-time pads $r_0, r_1 \in \{0,1\}^m$ and sends $y_0 = x_0 \oplus r_{b'}$ and $y_1 = x_1 \oplus r_{1-b'}$ to B.
3. A then computes t-out-of-t shares $(r_0^1, \ldots, r_0^t)$ and $(r_1^1, \ldots, r_1^t)$ of r_0, r_1, respectively.
4. For each $i \geq 3$, A sends shares r_0^i and r_1^i to party P_i.
5. For each $i \geq 3$, parties P_i and B invoke the OT protocol $\Pi^{G,1}_{P_i \to B}$ with inputs (r_0^i, r_1^i) and c respectively.
6. B uses the obtained shares $r_c^1, \ldots, r_c^t$ to reconstruct r_c.
7. B finally computes $y_b \oplus r_c = x_b$.

5.1 The t-claw Protocol

The first protocol we describe is the t-claw protocol, where the graph G describing the network is such that $G \simeq_{A,B} G^t_{\text{claw}}$. The protocol is described in Protocol 1. The protocol is a straightforward generalization of the one described in Section 3.2. The idea is for A to compute t-out-of-t shares of its inputs and distribute them among the t parties connected to B. These t parties then perform OT with B so that B receives the shares to reconstruct his output.

Lemma 5. *Protocol 1 is an efficient t-secure OT protocol for a network $G \simeq_{A,B} G^t_{\text{claw}}$ with A as the sender and B as the receiver.*

Proof Intuition. The t-security of the procotol can be seen as follows. Steps 1, 2 and 7 perform OT correction, that is, they perform a transformation from random OT to 1-out-of-2 OT. This transformation protects against the case that the parties $P_3, \ldots, P_{t+2}$ (that is, all but A and B) are corrupt. Suppose A were corrupt and B were honest. Clearly, A colluding with any of the parties $P_3, \ldots, P_{t+2}$ provides A with no additional information since all they possess are shares sent by A. Next, if A were honest and B corrupt, at least one of the parties $P_3, \ldots, P_{t+2}$ must be honest. B has no information about those shares and hence does not learn anything. Finally, if both A and

B were corrupt, there is nothing to prove.

Proof. Let $\mathcal{A}$ be a t-threshold adversary which corrupts parties T, $|T| \leq t$. We will construct a simulator $\mathcal{S}$ which plays the role of the uncorrupted parties. If $\{A, B\} \subset T$ then the uncorrupted parties receive no input, so $\mathcal{S}$ can perfectly simulate the uncorrupted parties. If $\{A, B\} \cap T = \emptyset$ then $\mathcal{S}$ chooses arbitrary inputs x_0, x_1, b and runs the protocol, invoking the OT simulator for each OT invocation with an uncorrupted party in step 5. Since corrupted parties only learn secret shares of independently random values, the view of the adversary is independent of the choice of x_0, x_1, b and is identical to the real world.

Otherwise, we have that the corrupted parties T include exactly one of A, B. If $A \in T$ but $B \notin T$, then $\mathcal{S}$ chooses arbitrary input b and runs the protocol, invoking the OT simulator for each OT invocation with an uncorrupted party in step 5. Since the OT simulator does not reveal the input c, and since the adversary only learns the direct sum of b with the random bit c, the view of the adversary is identical regardless of the value of b and in particular is identical to the real world.

Finally we have the case $B \in T, A \notin T$. Here the simulator $\mathcal{S}$ is given the output value x_b. $\mathcal{S}$ runs the protocol with (x_b, x_b) as the input to A, again invoking the OT simulator for each OT invocation with an uncorrupted party in step 5. Since $|T| \leq t$ and $B \in T$, at most $t-1$ of the t parties $P_3, \ldots, P_{t+2}$ are corrupted. Consequently the adversary observes at most $t-1$ shares of the random one-time pads r_0, r_1, so by the security of t-out-of-t secret sharing, conditioned on the remaining shares being hidden, the distribution of the observed shares is independent of r_0, r_1. The adversary learns the shares of r_c, but by the security of the OT channels, the view of the adversary in step 3 and onward is independent of the remaining shares of r_{1-c} and consequently is independent of the choice of r_{1-c}. Consequently the view of the adversary in step 3 and onward is independent of r_{1-c}. In step 2, the adversary sees $y_b = x_b \oplus r_c$ and $y_{1-b} = x_{1-b} \oplus r_{1-c}$, so by the security of the one-time pad, the view of the adversary is independent of x_{1-b}. Consequently the overall view of the adversary is identical in the real and ideal worlds. □

5.2 The t-clique Protocol

The next protocol we describe is the t-clique protocol, where the graph G describing the network is such that $G \simeq_{A,B} G^t_{\text{clique}}$. The protocol is described in Protocol 2. The protocol is a straightforward generalization of the one described in Section 3.3. The idea is for the parties $P_3, \ldots, P_{t+2}$ to compute t-out-of-t shares of OT correlations and send them to A and B respectively. The parties have a complete network of OT channels, so this can be done via multiparty computation (Lemma 2). A and B then perform OT correction using their secure channel.

Lemma 6. *Protocol 2 is an efficient t-secure OT protocol for a network $G \simeq_{A,B} G^t_{\text{clique}}$ with A as the sender and B as the receiver.*

Proof Intuition. The t-security of the procotol can be seen as follows. Steps 4, 5 and 6 perform OT correction, that is, they perform a transformation from random OT to 1-out-of-2 OT. This transformation protects against the case that all of parties $P_3, \ldots, P_{t+2}$ (that is, all but A and B) are corrupt. If one of A and B were corrupt, there exists at least one honest party among the parties $P_3, \ldots, P_{t+2}$. Hence, even by colluding, A or B would have no information about those shares and would not learn anything. Finally, if both A and B were corrupt, there is nothing to prove.

Protocol 2: t-clique Protocol

Preliminaries: Let $A, B, P_3, \dots, P_{t+2}$ be the $t+2$ parties in a network $G \simeq_{A,B} G^t_{\text{clique}}$. A has inputs $x_0, x_1 \in \{0,1\}^m$ and B has input $b \in \{0,1\}$.

Protocol:

1. Parties $P_3, \dots, P_{t+2}$ use their pairwise OT channels to run t-secure MPC for the function f using the protocol from Lemma 2 for the function f described ahead. The function f is to securely compute t-out-of-t shares $(r_0^1, \dots, r_0^t)$, $(r_1^1, \dots, r_1^t)$ of two randomly sampled one-time pad keys r_0, r_1, $(c^1, \dots, c^t)$ of a random bit $c \in \{0,1\}$, and independent shares $(s^1, \dots, s^t)$ of key r_c, so that party $i+2$ receives only shares r_0^i, r_1^i, s^i, c^i for each i.
2. Each party P_{i+2} for $i \geq 1$ sends shares r_0^i, r_1^i to A and s^i, c^i to B.
3. A uses shares $(r_0^1, \dots r_0^t)$ and $(r_1^1, \dots, r_1^t)$ to reconstruct r_0 and r_1.
4. B uses shares $(c^1, \dots, c^t)$ and $(s^1, \dots, s^t)$ to reconstruct c and r_c and sends $b' = b \oplus c$ to A.
5. A computes $y_0 = x_0 \oplus r_{b'}$ and $y_1 = x_1 \oplus r_{1-b'}$ and sends both to B.
6. B computes $y_b \oplus r_c = x_b$.

Proof. Let $\mathcal{A}$ be a t-threshold adversary which corrupts parties T, $|T| \leq t$. We will construct a simulator $\mathcal{S}$ which plays the role of the uncorrupted parties. As above, if $\{A, B\} \subset T$ then the uncorrupted parties receive no input, so $\mathcal{S}$ can perfectly simulate the uncorrupted parties. If $\{A, B\} \cap T = \emptyset$ then $\mathcal{S}$ chooses arbitrary inputs x_0, x_1, b and runs the protocol. Since the only steps which depend at all on the inputs are on point-to-point channels between A and B, the view of the adversary in the real and ideal worlds is identical.

Otherwise, we have that the corrupted parties T include exactly one of A, B. If $A \in T$ but $B \notin T$, then $\mathcal{S}$ chooses arbitrary input b and runs the protocol, invoking the MPC simulator for the protocol in step 1 (the existence of this simulator follows from Lemma 2). Since at least one of the parties $P_3, \dots, P_{t+2}$ is uncorrupted, the security of the MPC protocol implies that the view of the adversary is independent of the uncorrupted parties' shares s^i and c^i, and so by the security of the secret sharing scheme is independent of the value of the bit c. The only message received by the adversary which depends on b is the bit $b' = b \oplus c$, so it follows that the view of the adversary is independent of the bit b and therefore is identical in the real and ideal worlds.

Finally we have the case $B \in T, A \notin T$. Here the simulator $\mathcal{S}$ is given the output value x_b. $\mathcal{S}$ runs the protocol with (x_b, x_b) as the input to A, again invoking the MPC simulator for the protocol in step 1. Since at least one of the parties $P_3, \dots, P_{t+2}$ is uncorrupted, the security of the MPC protocol implies that the view of the adversary is independent of the uncorrupted parties' shares r_0^i and r_1^i, so by the security of t-out-of-t secret sharing, conditioned on the remaining shares being hidden, the distribution of the observed shares is independent of r_0, r_1. The adversary learns the shares of r_c, but by the security of the OT channels, the view of the adversary through step 4 is independent of the remaining shares of r_{1-c} and consequently is independent of the choice of r_{1-c}. In step 5, the adversary sees $y_b = x_b \oplus r_c$ and $y_{1-b} = x_{1-b} \oplus r_{1-c}$, so by the security of the one-time pad, the view of the adversary is independent of x_{1-b}. Consequently the overall view of the adversary is identical in the real and ideal worlds. □

5.3 Cascading

The following building block is a generalization of the technique described in Section 3.4. The technique describes a general method of combining protocols iteratively. In our context, this can be thought of a tool for transforming a network described by a graph G to one described by a graph G', where $G \subseteq_V G'$ and G and G' are both graphs on the same vertex set V. In other words, it describes protocols as adding new edges indicating the establishment of OT correlations between new pairs of parties in the network. With this abstraction, it is easy to view the technique of cascading as one which combines protocols iteratively to transform the underlying network by adding new edges. This is described formally below.

Definition 4. *Let $G = (V, E)$ and $G' = (V, E')$ be two graphs on the same set of vertices, V, with $G \subseteq_V G'$. We say that a protocol Π* t-transforms *a network G into the network G' if for each $\{P_i, P_j\} \in E' \setminus E$, Π is a t-secure OT protocol for a network G with P_i as the sender and P_j as the receiver.*[12]

Lemma 7. *If Π_1 is a protocol that runs in time T_1 and t-transforms network G_1 into G_2, and Π_2 is a protocol that runs in time T_2 and t-transforms network G_2 into G_3, then there exists a protocol Π that runs in time T_1T_2 and t-transforms G_1 into G_3.*

Proof. The protocol Π simply runs Π_2, running protocol Π_1 to obtain the necessary correlations whenever Π_2 invokes OT on an edge of $G_2 \setminus G_1$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the simulators associated with Π_1 and Π_2 respectively. The simulator for Π simply runs $\mathcal{S}_2$, invoking $\mathcal{S}_1$ for OT calls made on edges in $G_2 \setminus G_1$. □

Using OT extension [2, 43], we can also obtain a computationally secure version of cascading with improved efficiency.

Lemma 8. *Let λ be a computational security parameter. Assuming one-way functions or correlation-robust hash functions, if Π_1 is a protocol that runs in time T_1 and t-transforms network G_1 into G_2, and Π_2 is a protocol that runs in time T_2 and t-transforms network G_2 into G_3, then there exists a computationally secure protocol Π that runs in time $\lambda \cdot T_1 + T_2 \cdot \mathrm{poly}(\lambda)$ and t-transforms G_1 into G_3.*

Proof. First, run protocol Π_1 λ times on random inputs to obtain λ independent OT correlations for each edge of $G_2 \setminus G_1$. Then run Protocol Π_2, using OT extension [2, 43] to obtain OT correlations for OT calls made on edges in $G_2 \setminus G_1$. □

5.4 The 2-path graph

The protocol described in this section is a commonly used subroutine in several of the protocols which follow. It is a particular combination of the tools encountered in Sections 5.1, 5.2 and 5.3. The subroutine, which we call 2-path, is the same as the one described in Section 3.4. It is used to obtain OT correlations between parties who have a common neighbor in a four-party network with at most two corruptions (see Figure 4(c)).

[12] Note that a single protocol Π may set up independent random OT correlations for several pairs of parties $\{P_i, P_j\} \in E' \setminus E$. These correlations can be used to run 1-out-of-2 OT using OT correction.

Protocol 3: 2-path

Preliminaries: Let A, B, C, D be the parties, and let there exist OT channels (A, C) and (B, C). A has input (x_0, x_1), and B has input $b \in \{0, 1\}$.

Protocol:

1. Invoke Protocol 1 (2-claw) on parties (D, C, A, B) to obtain OT correlations on edge (D, C).
2. By Lemma 7, we have an OT channel between D and C.
3. Invoke Protocol 2 (2-clique) on parties (A, B, C, D).

Lemma 9. *Protocol 3 is an efficient* 2*-secure OT protocol for a network* $G \simeq_{A,B} G^2_{\text{2-path}}$ *with* A *as the sender and* B *as the receiver.*

Proof. This follows immediately from Lemma 7 and the 2-security of Protocols 1 and 2 for $t = 2$ (Lemmata 5 and 6). □

5.5 Combiners

OT combiners aim to combine several insecure candidate protocols for establishing OT correlations between two parties into a single secure protocol. For a class of adversaries $\mathbb{A}$, it is possible to achieve this when the candidate protocols satisfy the property that a majority of them are secure against each advesary $\mathcal{A} \in \mathbb{A}$. The following lemma is due to [56, 37], relying on prior work by [38, 65] and based on a construction by [22].

Lemma 10. *[56, 37] Let* $\mathbb{A}$ *be an adversary class. Suppose there exist* m *protocols* $\Pi_1, \ldots, \Pi_m$ *for* $f_{OT}(A, B, P_1, \ldots, P_n)$ *such that for any adversary* $\mathcal{A} \in \mathbb{A}$ *a majority of the protocols are secure. Then, there exists a protocol* $\Pi^*(\Pi_1, \ldots, \Pi_m)$ *for* $f_{OT}(A, B, P_1, \ldots, P_n)$ *which is secure against all adversaries* $\mathcal{A} \in \mathbb{A}$*. Moreover, if each protocol* Π_i *is efficient and perfectly secure, then so is* Π^*.

6 The case $t = n/2$

We now consider the specific case of $t = n/2$, that is, when at most half the parties are corrupt. We note that this is the smallest value of t for which the question is non-trivial. From the lower bounds proven in Theorem 4, we already have that for all n-party networks G containing A and B such that $G \subseteq_{A,B} \Lambda^0_{n/2}$, there exists no $n/2$-secure OT protocol with A as the sender and B as the receiver. Surprisingly Theorem 11 shows that these are the only networks for which $(n/2)$-secure OT between A and B is impossible. Below, we provide an explicit $n/2$-secure OT protocol between A and B whenever the network G is $(n/2, A, B)$-unsplittable.

Theorem 11. *Let* G *be an* n*-party network OT containing parties* A *and* B*. Then Protocol 5 is an* $n/2$*-secure OT protocol between* A *and* B *if and only if* G *is* $(n/2, A, B)$*-unsplittable.*

We analyze the efficiency of the protocol in Theorem 13 below. The protocol as stated runs in quasi-polynomial time. We can also obtain a computationally secure protocol which runs in polynomial time. The protocol we describe proceeds in two stages. In the first stage, the protocol transforms every connected component of the network into a clique. This transformation is very

Protocol 4: Completing Triangles

Preliminaries: Let $A, B, C, P_4, \dots, P_n$ be the n parties, and let there exist OT channels (A, C) and (B, C). A has input (x_0, x_1), and B has input $b \in \{0, 1\}$.

Protocol:

1. Run a combined protocol $\Pi^*(\Pi_4, \dots, \Pi_n)$ on the $n-3$ protocols $\Pi_4, \dots, \Pi_n$, where
   - For each $i \geq 4$, Π_i denotes an invocation of Protocol 3 (2-path) with the four parties A, B, C, P_i with A as the sender and B as the receiver.

specific to the case of $t = n/2$, and in particular, for $t > n/2$ a connected component cannot in general function as a clique. This transformation is carried out by means of repeatedly calling Protocol 4, which obtains OT correlations between a pair of parties who have a common neighbour. This protocol uses the building block Protocol 3 from Section 5.4 along with machinery of OT combiners described in Section 5.5.

Lemma 12. *Let G be an n-vertex OT network with edges $\{A, C\}$ and $\{B, C\}$. Protocol 4 is an $n/2$-secure OT protocol for the network G with A as the sender and B as the receiver.*

Proof. We consider cases depending on the number of corrupted parties in the set $T = \{A, B, C\}$. If T contains at most one corrupted party, then each tuple (A, B, C, P_i) for $i \geq 4$ contains at most 2 corrupted parties, so each protocol Π_i in step 1 is secure. If T contains two corrupted parties, then there are at most $t - 2 = (n-4)/2$ corrupted parties among $P_4, \dots, P_n$, so a majority of these parties are honest. Consequently a majority of the protocols Π_i which are combined in step 1 are secure. Thus, in either case, by Lemma 10 the protocol is secure. Finally, if all three parties of T are corrupted, then all uncorrupted parties receive no input, so the simulator $\mathcal{S}$ can perfectly simulate the uncorrupted parties by running the honest protocol. Therefore Protocol 4 is $n/2$-secure. □

We now complete the proof of Theorem 11.

Proof Intuition (Theorem 11): It is easy to see that by invoking Protocol 4 repeatedly, one can obtain OT correlations between any pair of parties in the same connected component. In other words, using cascading (Lemma 7), we can assume that we are given a network which consists of disjoint cliques. This is done in step 1 of Protocol 5. Hence, if A and B were in the same connected component in G, this process would end up with correlations between A and B and we can terminate the protocol (step 2).

If A and B are in different components, then a natural next step is to run the clique protocol described in Section 5.2 with each of the cliques and parties A and B with the intent of setting up OT correlations between A and B. However, the number of corruptions t may be greater than the size of any clique, and so Protocol 2 may not be secure. However, for an invocation to be secure, we only require that the clique contains at least one honest party. A majority of parties must be in cliques containing at least one honest party, so if we invoke Protocol 2 for each of the parties on their respective cliques, for any adversary a majority of the invocations is secure. By Lemma 10 we can combine these candidate protocols to obtain a single secure protocol. This is performed in step 5 of Protocol 5. Finally, we note that steps 3, 4 and 6 perform OT correction, that is, they perform a transformation from random OT to 1-out-of-2 OT. This yields the $n/2$-security of Protocol 5.

Protocol 5: $n/2$ corruptions

Preliminaries: Let $P_1 = A, P_2 = B, P_3, \dots, P_n$ be the n parties in a network $G = (V, E)$. A has input (x_0, x_1), and B has input $b \in \{0, 1\}$.

Protocol:

1. While there exist parties $P_i, P_j, P_k \in V$ such that $\{P_i, P_j\} \in E$, $\{P_j, P_k\} \in E$, but $\{P_i, P_k\} \notin E$:
   (a) Let S be the set of triples of distinct vertices $(X, Y, Z) \in V^3$ with $\{X, Y\} \in E, \{Y, Z\} \in E$, and $\{X, Z\} \notin E$.
   (b) For each triple $(X, Y, Z) \in S$, invoke Protocol 4 with independent random inputs $(r_0^{i,k}, r_1^{i,k})$ and $b^{i,k}$, to obtain OT correlations along edge $\{X, Z\}$.
   (c) Invoking cascading (Lemma 7), we can add $\{X, Z\}$ to the edge set E for all triples $(X, Y, Z) \in S$.

   The OT network G now consists of disjoint cliques $\mathcal{C}_1, \dots, \mathcal{C}_\ell$.
2. If A and B are in the same clique, then halt.
3. B samples a random bit c and sends $b' = b \oplus c$ to A.
4. A chooses random one-time pads r_0, r_1 and sends $y_0 = x_0 \oplus r_{b'}$ and $y_1 = x_1 \oplus r_{1-b'}$ to B.
5. Let $\mathcal{C}_1$ be the clique containing A and $\mathcal{C}_2$ be the clique containing B. For each party P_i, $i \geq 3$, let $\mathcal{C}(i)$ denote the clique containing party i, and let $P_{j_1}, \dots, P_{j_{|\mathcal{C}(i)|}}$ denote the parties in clique $\mathcal{C}(i)$.
   Run a combined protocol $\Pi^*(\Pi_1, \dots, \Pi_n)$ on the n protocols $\Pi_1, \dots, \Pi_n$, where
   - For each $i \in [n]$, Π_i denotes an invocation of Protocol 2 on the $|\mathcal{C}(i)|+2$ parties $A, B, P_{j_1}, \dots, P_{j_{|\mathcal{C}(i)|}}$ with inputs (r_0, r_1) and c.[a]
6. Finally, B computes $x_b = y_b \oplus r_c$.

[a] In the case $\mathcal{C}(i) = \mathcal{C}_1$, A is both the OT sender and a member of the clique. A similar condition holds for B in the case $\mathcal{C}(i) = \mathcal{C}_2$.

Proof of Theorem 11. The "only if" part of theorem has been proven by virtue of the lower bound of Theorem 4 with $t = n/2$. We now prove the "if" part. We note that in the case where A and B are in the same connected component in the network G, by the $n/2$-security of Protocol 4 and Lemma 7, we note that Protocol 5 is an $n/2$-secure OT protocol with A as the sender and B as the receiver, thus proving the theorem.

We now proceed to the case where A and B are not in the same connected component in G. We must show that the protocol is secure against t-threshold adversaries as long as the vertices cannot be partitioned into two sets V_A, V_B each of size $t = n/2$ with $A \in V_A, B \in V_B$ such that there are no edges between V_A and V_B. Let $\mathcal{A}$ be a t-threshold adversary which corrupts parties T, $|T| \leq t$. We will construct a simulator $\mathcal{S}$ which plays the role of the uncorrupted parties.

If $\{A, B\} \subset T$ then the uncorrupted parties receive no input, so the simulator can perfectly simulate the uncorrupted parties. If $\{A, B\} \cap T = \emptyset$ then $\mathcal{S}$ chooses arbitrary inputs x_0, x_1, b and runs the protocol. Since the only steps which depend on the input at all are on point-to-point channels between A and B, the view of the adversary in the real and ideal worlds is identical.

Otherwise, we have that the corrupted parties T include exactly one of A, B. If $A \in T$ but $B \notin T$, then $\mathcal{S}$ chooses an arbitrary bit b and runs the protocol, invoking the OT simulator for each invocation of Protocol 4. It follows that as long as the combined protocol Π^* in step 5 is secure against $\mathcal{A}$, Protocol 5 is secure against $\mathcal{A}$. It remains to show that a majority of the n protocols $\Pi_1, \dots, \Pi_n$ are secure against $\mathcal{A}$. Since party B is honest, by Lemma 6, protocol Π_i is secure against

$\mathcal{A}$ as long as at least one of the parties in clique $\mathcal{C}(i)$ is honest. In particular, if party P_i is honest then protocol Π_i is secure against $\mathcal{A}$. At most t of the parties $P_1, \ldots, P_n$ are corrupt, so the only protocols which may be insecure against $\mathcal{A}$ are the t protocols Π_i corresponding to the corrupted parties P_i. Assume that all t of these protocols are insecure against $\mathcal{A}$. Then the corrupted parties lie in completely corrupted cliques who sizes sum to $n/2$. This then gives a set $V_A = T$ of $n/2$ parties containing A but not B such that there are no edges from V_A to the remaining vertices $V_B = \overline{T}$. However, we know that G possesses no such partition. Hence, at most $t - 1 < n/2$ of the n protocols are insecure against $\mathcal{A}$ and hence by Lemma 10, the combined protocol Π^* in step 5 is secure and hence Protocol 5 is secure against $\mathcal{A}$.

The remaining case that $B \in T$ but $A \notin T$ is similar. Here, the simulator $\mathcal{S}$ is given the output value x_b. $\mathcal{S}$ runs the protocol with (x_b, x_b) as the input to A, again invoking the OT simulator for each invocation of Protocol 4. As above, as long as the combined protocol Π^* in step 5 is secure against $\mathcal{A}$, Protocol 5 is secure against $\mathcal{A}$. By the same argument, the only protocols Π_i which may be insecure against $\mathcal{A}$ are the t protocols corresponding to the corrupted parties P_i. If all t of these protocols are insecure against $\mathcal{A}$, we have a set $V_A = \overline{T}$ of $n/2$ parties containing A but not B such that there are no edges from V_A to the remaining vertices $V_B = T$. However, we know that G possesses no such partition, so at most $t - 1 < n/2$ of the n protocols are insecure against $\mathcal{A}$. By Lemma 10, the combined protocol Π^* in step 5 is secure and so Protocol 5 is secure against $\mathcal{A}$. □

We now analyze the efficiency of Protocol 5.

Theorem 13. *Protocol 5 runs in quasi-polynomial time. Assuming one-way functions, we can obtain a computationally secure protocol which runs in polynomial time using computationally secure cascading (Lemma 8).*

Proof. Each iteration of step 1 decreases the length of a path between any pair of vertices from ℓ to $\lceil \ell + 1 \rceil / 2$. Consequently, after $O(\log n)$ iterations the graph will consist of a collection of disjoint cliques, and the protocol will move on to the next step. By Lemma 7 (Cascading), if each iteration can be performed in time at most T assuming the augmented graph, then the full cascaded protocol runs in time at most $T^{O(\log n)}$. Since $T = \mathrm{poly}(n)$ and each other step of the protocol is efficient, this implies that Protocol 5 runs in quasi-polynomial time.

Replacing the cascading of step 1 with the more efficient but computationally secure cascading of Lemma 8, we have the cascaded protocol runs in time $O(T \,\mathrm{poly}(\lambda) \cdot \log n)$. Since each other step of the protocol is efficient, this implies that assuming one-way functions, we have a computationally-secure version of Protocol 5 that runs in quasi-polynomial time. □

7 The case $t = n - 2$

On account of the lower bound proven in [36], we note that $t = n - 2$ is the largest value of t for which the question is non-trivial. In this section we present an improved computationally efficient OT protocol between A and B for the special case $t = n - 2$ for all $(2, A, B)$-unsplittable networks G.

Theorem 14. *Let G be an n-party OT network containing parties A and B. Then Protocol 6 is an efficient $(n - 2)$-secure OT protocol between A and B if and only if G is $(2, A, B)$-unsplittable.*

Protocol 6: $n-2$ **corruptions**

Preliminaries: Let $P_1 = A, P_2 = B, P_3, \ldots, P_n$ be the n parties, and let graph $G = (V, E)$ be the OT network among the parties. A has input (x_0, x_1), and B has input $b \in \{0, 1\}$.

Protocol:

1. For all pairs of parties $P_i, P_j \in V$ with $i, j \geq 3$ such that $\{P_i, P_j\} \notin E$:
   (a) Invoke Protocol 5 (or any 2-secure protocol for $n' = 4$) on the induced OT subgraph $G_{i,j} := G \cap \{P_i, P_j, A, B\}$ with independent random inputs $(r_0^{i,j}, r_1^{i,j})$ and $b^{i,j}$, to obtain OT correlations along edge $\{P_i, P_j\}$.
   (b) By virtue of cascading (Lemma 7), we can add edge $\{P_i, P_j\}$ to the graph G.[a]

   The OT network G now contains a $(n-2)$-clique among vertices $P_i, \ldots, P_n$.
2. Invoke Protocol 2 (t-clique) with input (x_0, x_1) and b.

[a] We will only have OT security over this edge when at least two of the parties P_i, P_j, A, B are honest, but we obtain the functionality of the edge regardless. We address security of the overall protocol in the proof.

The protocol is built upon the following structural aspect of the network G under consideration. Since G is $(2, A, B)$-unsplittable, for any two sets of vertices V_AA and V_BB such that $|V_A| = |V_B| = 2$, there exists an edge from a vertex of V_A to a vertex of V_B. In particular, this implies that for any two parties P_i, P_j where $i, j \geq 3$, the sub-network $G_{i,j}$ induced by parties A, B, P_i and P_j is $(2, A, B)$-unsplittable. Then for any i, j, we also have that the sub-network $G_{i,j}$ is $(2, P_i, P_j)$-unsplittable. Hence, we could try to obtain OT correlations between every pair of vertices P_i, P_j by running Protocol 5 on every $G_{i,j}$ for $n = 4$ parties. Notice that if these invocations were secure, then we would obtain an $(n-2)$-clique in the network after which we can execute Protocol 2 in order to obtain OT correlations between A and B. This is described in Protocol 6. However, each of the execution of Protocol 5 is only guaranteed to be secure if at most two of the corresponding parties are corrupt. This need not be true in general, and so we cannot directly leverage the security of Protocol 5. Nonetheless, we will argue that Protocol 6 is secure against $t = n-2$ corruptions.

Proof Intuition (Theorem 14): In order to analyze the $(n-2)$-security of Protocol 6, we consider each invocation of Protocol 5 on a sub-network $G_{i,j}$. If at most two of the four parties in $G_{i,j}$ are corrupt, then that invocation of Protocol 5 is secure and yields secure OT correlations between parties P_i and P_j. Appealing to Lemma 7, we can augment G to include edge $\{P_i, P_j\}$.

Each $G_{i,j}$ must contain at least one honest party since either A or B must be honest (otherwise, there is nothing to prove). It remains to consider sub-networks $G_{i,j}$ in which three of the parties are corrupt. Since at least one of A or B is honest, this implies that both P_i and P_j are corrupt. Thus, there is nothing to prove regarding the security of the invocation of Protocol 5 on $G_{i,j}$ since we are establishing OT correlations between a pair of corrupt parties P_i and P_j. Combining these claims, we have that each of the invocations of Protocol 5 is secure and yields secure OT correlations between the pairs of parties P_i, P_j for all $i, j \geq 3$. By virtue of Lemma 7, we obtain an $(n-2)$-clique in the network and the $(n-2)$-security of Protocol 2 with $t = n-2$ proves the $(n-2)$-security of Protocol 6.

Proof. The "only if" part of theorem follows from the lower bound of Theorem 4 with $t = n-2$. We now prove the "if" part. Let $\mathcal{A}$ be a t-threshold adversary which corrupts parties T, $|T| \leq t = n-2$.

If A and B are both corrupt then the uncorrupted parties receive no input, so the simulator $\mathcal{S}$ can perfectly simulate the uncorrupted parties.

Otherwise we have that at least one of A or B is uncorrupted. We first show how to simulate step 1. Since G is $(2, A, B)$-unsplittable, for each pair of parties P_i, P_j considered in step 1, we have that $G_{i,j}$ is $(2, A, B)$-unsplittable. Since $\{P_i, P_j\} \notin E$, this implies that some vertex among A, B, P_i, and P_j has degree at least 2 in $G_{i,j}$. In particular, we have $G_{i,j} \not\subseteq_{P_i,P_j} \Lambda_2^0$, where Λ_2^0 is labeled so that P_i and P_j are in separate components. Consequently, by Theorem 11, we securely obtain OT correlations between P_i and P_j whenever at most two of the parties A, B, P_i, P_j are corrupt.

Since A and B are not both corrupt, if P_i and P_j are also not both corrupt, then at most two of the four parties A, B, P_i, P_j are corrupt. Consequently we securely obtain OT correlations between P_i and P_j. Otherwise, P_i and P_j are both corrupt, in which case there is nothing to prove. More formally, there is a simulator which can perfectly simulate the invocation of Protocol 5 in step 1a by executing the protocol of the honest parties, since the honest parties receive no input. Consequently we securely obtain OT correlations between the corrupted parties P_i and P_j. In both cases, by Lemma 7, we have an OT simulator which can simulate subsequent invocations of the OT channel between P_i and P_j.

Therefore, by the end of step 1 we t-securely obtained an OT channel between every pair of parties P_i, P_j for $i, j \geq 3$. Then by Lemma 6, the invocation of Protocol 2 in step 2 is a t-secure OT protocol with A as the sender and B as the receiver, as desired. □

8 The General Case: $t \geq n/2$

In this section, we resolve the network OT question for general $t \geq n/2$. Note that from the protocols in Sections 6 and 7 we already have tight answers for the special cases $t = n/2$ and $t = n - 2$. We address the general question from both ends of the spectrum, namely for t larger than $n/2$ and t smaller than $n - 2$. These analyses yield two distinct protocols which employ the protocols from Sections 6 and 7 as their respective base cases. The two protocols we describe are efficient in different parameter regimes. Protocol 7 described in Section 8.1 is quasi-polynomially efficient[13] when $t = n/2 + \mathcal{O}(1)$, and Protocol 8 described in Section 8.2 is (polynomially) efficient when $t = n - \mathcal{O}(1)$. Putting these protocols together, we obtain a single protocol that is efficient under computational security when either $t = n/2 + \mathcal{O}(1)$ or $t = n - \mathcal{O}(1)$. We note that the problem of recognizing whether there exists a t-secure OT protocol is efficient in these cases, while the recognition problem for general n, t is coNP-complete.

8.1 General Protocol (Quasi-polynomial for $t = n/2 + \mathcal{O}(1)$)

We now describe a t-secure OT protocol between A and B for all $(n-t, A, B)$-unsplittable networks G. As a consequence of the lower bound described in Section 4, this result is tight.

Theorem 15. *Let G be an n-party OT network containing parties A and B, and let $t \geq n/2$. Then Protocol 7 is a t-secure OT protocol between A and B if and only if G is $(n - t, A, B)$-unsplittable. The protocol achieves perfect security and runs in quasi-polynomial time for $t = n/2 + \mathcal{O}(1)$. Assuming one-way functions, we can also obtain a protocol which achieves computational security and runs in polynomial time for $t = n/2 + \mathcal{O}(1)$.*

[13] or polynomially efficient under computational security

Protocol 7: General Protocol I

Preliminaries: Let $A, B, P_3, \ldots, P_n$ be the n parties in a network G and let $t \geq n/2$ be the maximum number of corruptions. A has input (x_0, x_1), and B has input $b \in \{0, 1\}$.

Protocol:

1. If $t = n/2$, then invoke Protocol 5 and halt.
2. Otherwise, run a combined protocol $\Pi^*(\Pi_3, \ldots, \Pi_n)$, where
   - For each $i \geq 3$, Π_i denotes the recursive invocation of this protocol on the $n-1$ parties excluding party P_i with the induced sub-network $G \setminus \{P_i\}$ and $t' = t - 1$ corruptions.

The protocol proceeds by recursion, reducing the problem of obtaining an OT protocol on an n-vertex graph with $t > n/2$ corrupted parties to a number of instances of n'-vertex graphs, a majority of which have at most t' corrupted parties, for $n' = n - 1$ and $t' = t - 1$. As shown below, each n'-vertex subgraph G' has a structure similar to G in the sense that G' is $(n' - t', A, B)$-unsplittable whenever G is $(n - t, A, B)$-unsplittable. We can now recurse on these smaller problem instances, invoking an OT combiner to obtain the full protocol.

More precisely, the protocol constructs $n - 2$ subgraphs on $n - 1$ vertices, where each subgraph is obtained by deleting a single vertex other than A and B. We can recursively run a $(t-1)$-secure OT protocol on each of the subgraphs. The final protocol invokes a combiner on these $n - 2$ candidate protocols. It remains to be shown that a majority of the subgraphs G' contain at most $t - 1$ corrupt parties.

Proof Intuition (Theorem 15): We may assume that at least one of A or B is honest. As described above, we wish to argue that a majority of the subgraphs G' contain at most $t - 1$ corrupt parties. Combining this with the claim that these subgraphs preserve an unsplittability property of G and invoking Lemma 10 completes the proof.

However, this claim follows from the following observation. Since $t > n/2$, if exactly t parties are corrupt then a majority of the subgraphs contain at most $t - 1$ corrupt parties since A and B are not both corrupt. If strictly fewer than t parties are corrupt then all of the sub-graphs contain at most $t - 1$ corrupt parties. In either case, for a majority of subgraphs, at most $t - 1$ of the parties are corrupt.

We first present and prove a structure lemma.

Lemma 16. *Given graph $G = (V, E)$ and a vertex i, let G_i be the induced graph on the $n - 1$ vertices $V \setminus \{i\}$. If G is $(n - t, A, B)$-unsplittable, then G_i is also $(n - t, A, B)$-unsplittable.*

Proof. We will prove the contrapositive. Suppose that $G_i \subseteq_{A,B} \Lambda_{n-t}^{2t-n-1}$. This means there exists a partition of the vertex set of G_i as $V \setminus \{i\} = V_A \dot{\bigcup} V_S \dot{\bigcup} V_B$ with no edges between V_A and V_B, where $A \in V_A$, $B \in V_B$, $|V_A| = |V_B| = n - t$ and $|V_S| = 2t - n - 1$. But then we can partition the vertex set of G as $V = V_A \dot{\bigcup} V'_S \dot{\bigcup} V_B$, where $V'_S = V_S \cup \{i\}$. We have that $|V_A| = |V_B| = n - t$ and $|V'_S| = 2t - n$, and there are no edges between V_A and V_B, so $G \subseteq_{A,B} \Lambda_{n-t}^{2t-n}$, which is a contradiction. □

Using the lemma, we now prove Theorem 15.

Proof of Theorem 15: The "only if" direction follows from the lower bound of Theorem 4, and the efficiency claim follows immediately from Theorem 13. We now prove the "if" direction by induction on $2t - n$. The base case of $2t - n = 0$ follows from Theorem 11. Now assume for induction that the statement holds for $n' = n - 1, t' = t - 1$ with $t > n/2$.

Let $\mathcal{A}$ be a t-threshold adversary which corrupts parties T, $|T| \leq t$. If $\{A, B\} \subset T$ then the uncorrupted parties receive no input, so the simulator can perfectly simulate the uncorrupted parties. Consequently it suffices to consider the case $\{A, B\} \cap T \leq 1$. By Lemma 16, each sub-network $G \setminus \{P_i\}$ is $(n' - t', A, B)$-unsplittable. For each $i \geq 3$, let $T_i = T \setminus \{P_i\}$ denote the corrupted parties in sub-network $G \setminus \{P_i\}$. If $|T| < t$ then $|T_i| \leq |T| \leq t'$ for all i. Otherwise, $|T| = t$. In this case, since $\{A, B\} \cap T \leq 1$, it follows that at least $t - 1$ of the $n - 2$ parties $P_3, \ldots, P_n$ are corrupted. Since $t > n/2$, this is a majority of the parties $P_3, \ldots, P_n$. For each corrupted party P_i, $|T_i| = t'$. Consequently in each case we have that a majority of the sets $T_3, \ldots, T_n$ satisfy $|T_i| \leq t'$. Therefore by our inductive assumption, a majority of the protocols $\Pi_3, \ldots, \Pi_n$ in step 2 are secure against $\mathcal{A}$. By Lemma 10, we have that the combined protocol Π^* is secure against $\mathcal{A}$, so by induction, the theorem holds for all n, t with $t \geq n/2$. □

As an immediate consequence, the condition described in Theorem 15 is both necessary and sufficient in order to obtain a complete network of OT channels and perform secure multiparty computation among all parties in the network.

Corollary 17. *Let G be an n-party network. For $t \geq n/2$, we can t-securely generate OT correlations between all pairs of parties (thus, completing the OT network) if and only if the G is $(n - t)$-unsplittable.*

Proof. If G is $(n - t)$-unsplittable, then for all pairs of vertices A, B, G is $(n - t, A, B)$-unsplittable and hence from Theorem 15, we can generate OT correlations between A and B. Conversely, if $G \subseteq \Lambda_n^{2t-n}$, then choosing vertics $A \in V_A$ and $B \in V_B$, we have that $G \subseteq_{A,B} \Lambda_{n-t}^{2t-n}$ and hence Theorem 15 rules out the existence of such a protocol. Hence, there exist a pair of vertices between whom we cannot t-securely generate OT correlations. This completes the proof. □

8.2 General Protocol (Efficient for $t = n - \mathcal{O}(1)$)

We now describe another t-secure OT protocol for all networks G with A as the sender and B as the receiver whenever the network G is $(n - t, A, B)$-unsplittable. This protocol uses, in spirit, a reduction in the opposite sense than the one described in Section 8.1. The protocol is efficient whenever $t = n - \mathcal{O}(1)$.

Theorem 18. *Let G be an n-party OT network containing parties A and B, and let $t \geq n/2$. Protocol 8 is a t-secure OT protocol between A and B if and only if G is $(n - t, A, B)$-unsplittable. The protocol is efficient for $t = n - \mathcal{O}(1)$.*

The idea behind this protocol is the following. We increase the size of the network in order to obtain a large number N of well-connected additional simulated parties such that at least one them is guaranteed to be honest. We may assume that at least one of A and B is honest, as otherwise there is nothing to prove. Consequently there are at least two honest parties in the augmented network. We will now apply the protocol from Section 7. It remains to describe the construction of these simulated parties, to show that at least one of them is honest, and to prove a structural lemma that if the original network G is $(n - t, A, B)$-unsplittable then the augmented

Protocol 8: General protocol II

Preliminaries: Let $P_1 = A, P_2 = B, P_3, \ldots, P_n$ be the n parties in a network $G = (V, E)$. A has input (x_0, x_1), and B has input $b \in \{0, 1\}$. Let $k = n - t$.

Protocol:

1. Invoke Protocol 6 with $t' = n-2$ on the n'-node network G' with inputs (x_0, x_1) and b, where $n' = \binom{n-2}{k-1}+2$, and
   - S_{k-1} is the set of subsets of $\{P_3, \ldots, P_n\}$ of size $k - 1$.
   - The n' vertices of G' correspond to A, B, and the $\binom{n-2}{k-1}$ subsets of S_{k-1}.
   - The edges of G' are defined as follows. Two subsets $X, Y \in S_{k-1}$ will have an edge if either $X \cap Y \neq \emptyset$ or there exists a pair of parties $P_i \in X$ and $P_j \in Y$ with $\{P_i, P_j\} \in E$.
   - Invocation of OT over an edge $\{X, Y\}$ in G' with inputs (z_0, z_1) and c is performed as follows.
     - If $X \cap Y \neq \emptyset$, then choose some party $P_i \in X \cap Y$. $P_i \in X$ and hence knows (z_0, z_1); similarly, $P_i \in Y$ and knows c. Consequently P_i knows z_c, and sends it to the other members of set Y.
     - If $X \cap Y = \emptyset$, there is a pair of parties $P_i \in X, P_j \in Y$ such that $\{P_i, P_j\} \in E$. P_i knows (z_0, z_1) and P_j knows c, so they can invoke OT over the channel (P_i, P_j) in G, and P_j can then send the value z_c to the other members of set Y.

network G' is $(2, A, B)$-unsplittable.

Proof Intuition (Theorem 18): We first describe the new network generated by Protocol 8. The parties other than A and B in the newly constructed network consist of all subsets of size $n - t - 1$ of the parties in G containing neither A nor B. Lemma 19 below shows that this new network G' is $(2, A, B)$-unsplittable whenever G is $(n - t, A, B)$-unsplittable, where the edges of G' are as described in Protocol 8. A party X in G' will be considered honest if all constituent parties $P_i \in X$ from G are honest. Since one of A and B is honest and at most t parties are corrupt, at least $n - t$ parties are honest and in particular, at least $n - t - 1$ of the parties other than A and B must be honest. This means that one of the subsets is completely honest. Since A or B is also honest, G' is guaranteed to have at least two honest parties. Combining these facts and invoking Theorem 14 completes the argument.

We will use the following structural lemma about the network G' constructed in Protocol 8.

Lemma 19. *If G is $(n - t, A, B)$-unsplittable, then G' is a $(2, A, B)$-unsplittable network on $n' = \binom{n-2}{n-t-1} + 2$ vertices, where G' is the network produced in Protocol 8.*

Proof. We prove the contrapositive. Assume that $G' \subseteq_{A,B} \Lambda_2^{n'-2}$. Let $k = n - t$, and for $i \in \mathbb{N}$, let S_i denote the set of subsets of $V \setminus \{A, B\} = \{P_3, \ldots, P_n\}$ of size i. Then there exist vertices $X, Y \in S_{k-1}$ such that there are no edges in G' between any of the parties in $\{A, X\}$ and any of the parties in $\{B, Y\}$. In particular, $X \cap Y = \emptyset$, since otherwise $\{X, Y\}$ would be an edge of G'. This implies that we have $2k = 2(n - t)$ parties $\{A, B\} \cup X \cup Y$ such that there are no edges in G from the $n - t$ parties $\{A\} \cup X$ to any of the $n - t$ parties $\{B\} \cup Y$. By definition, this means that $G \subseteq_{A,B} \Lambda_{n-t}^{2t-n}$, which is a contradiction. □

Using this lemma, we will now prove the correctness of Protocol 8.

Proof of Theorem 18. As before, we only need to prove the "if" part. Let $\mathcal{A}$ be a t-threshold adversary which corrupts parties T, $|T| \leq t$. If A and B are both corrupt, then the honest parties have no input, so the simulator $\mathcal{S}$ can perfectly simulate the uncorrupted parties. If A and B are both honest, then $\mathcal{S}$ chooses arbitrary inputs x_0, x_1, b and runs the protocol. Since the only steps which depend at all on the inputs are on point-to-point channels between A and B, the view of the adversary in the real and ideal worlds is identical.

Otherwise, we have that at most $t-1$ of the parties $P_3, \ldots, P_n$ are corrupt and that either A or B is honest. In particular, for $k = n - t$, there are at least $k-1$ uncorrupted parties among $P_3, \ldots, P_n$. Consequently, S_{k-1} contains some set X consisting only of honest parties. We treat a party X in G' as honest if all constituent parties $P_i \in X$ are honest. Since A and B are not both corrupt, we have that G' contains at least two honest parties. By Lemma 19, G' is $(2, A, B)$-unsplittable. Consequently by Theorem 14 there is a simulator $\mathcal{S}'$ which simulates the roles of the honest parties in the invocation of Protocol 6 on G' in step 1. We define a simulator $\mathcal{S}$ for Protocol 8 which behaves exactly as an honest party for communication within each party $X \in S_{k-1}$ and invokes $\mathcal{S}'$ for any communication between parties in G'. The behavior of this protocol is identical to the behavior of $\mathcal{S}'$. Hence, by the correctness of simulator $\mathcal{S}'$, we have that the view of the adversary is identical in the real and ideal worlds. □

References

[1] B. Applebaum, Y. Ishai, E. Kushilevitz, and B. Waters. Encoding functions with constant online rate or how to compress garbled circuits keys. In *Crypto (2)*, pages 166–184, 2013.

[2] D. Beaver. Correlated pseudorandomness and the complexity of private computations. In *STOC*, pages 479–488, 1996.

[3] D. Beaver. Precomputing oblivious transfer. In *Crypto*, pages 97–109, 1995.

[4] Z. Beerliová-Trubíniová and M. Hirt. Perfectly-secure MPC with linear communication complexity. In *TCC*, pages 213–230, 2008.

[5] A. Beimel, T. Malkin, and S. Micali. The all-or-nothing nature of two-party secure computation. In *Crypto*, pages 80–97, 1999.

[6] A. Beimel, E. Omri, and I. Orlov. Protocols for multiparty coin toss with dishonest majority. In *Crypto*, pages 538–557, 2010.

[7] M. Bellare, V.T. Hoang, S. Keelveedhi, and P. Rogaway. Efficient garbling from a fixed-key blockcipher. In *IEEE Security and Privacy*, pages 478–492, 2013.

[8] M. Ben-Or, S. Goldwasser, and A. Wigderson. Completeness theorems for noncryptographic fault-tolerant distributed computations. In *STOC*, pages 1–10, 1988.

[9] E. Ben-Sasson, S. Fehr, and R. Ostrovsky. Near-linear unconditionally-secure multiparty computation with a dishonest minority. In *Crypto*, pages 663–680, 2012.

[10] R. Bendlin, I. Damgård, C. Orlandi, and S. Zakarias. Semi-homomorphic encryption and multiparty computation. In *Eurocrypt*, pages 169–188, 2011.

[11] D. Bogdanov, S. Laur, and J. Willemson. Sharemind: A framework for fast privacy-preserving computations. In *ESORICS*, pages 192–206, 2008.

[12] P. Bogetoft, D. Christensen, I. Damgård, M. Geisler, T. Jakobsen, M. Krøigaard, J. Nielsen, J. Nielsen, K. Nielsen, J. Pagter, M. Schwartzbach, and T. Toft. Secure multiparty computation goes live. In *FC*, pages 325–343, 2009.

[13] E. Boyle, K-M. Chung, and R. Pass. Large-scale secure computation: Multi-party computation for (parallel) ram programs. In *Crypto (2)*, pages 742–762, 2015.

[14] E. Boyle, S. Goldwasser, and S. Tessaro. Communication locality in secure multi-party computation - how to run sublinear algorithms in a distributed setting. In *TCC*, pages 356–376, 2013.

[15] R. Canetti, I. Damgård, S. Dziembowski, Y. Ishai and T Malkin. On adaptive vs. non-adaptive security of multiparty protocols. In *Eurocrypt*, pages 262–279, 2001.

[16] D. Chaum, C. Crépeau, and I. Damgård. Multiparty unconditionally secure protocols. In *STOC*, pages 11–19, 1988.

[17] B. Chor and E. Kushilevitz. A zero-one law for boolean privacy (extended abstract). In *STOC*, pages 62–72, 1989.

[18] T. Chou and C. Orlandi. The simplest protocol for oblivious transfer. In *Latincrypt*, pages 40–58, 2015.

[19] I. Damgård and Y. Ishai. Scalable secure multiparty computation. In *Crypto*, pages 501–520, 2006.

[20] I. Damgård, Y. Ishai, and M. Krøigaard. Perfectly secure multiparty computation and the computational overhead of cryptography. In *Eurocrypt*, pages 445–465, 2010.

[21] I. Damgård, Y. Ishai, M. Krøigaard, J. Nielsen, A. Smith. Scalable multiparty computation with nearly optimal work and resilience. In *Crypto*, pages 241–261, 2008.

[22] I. Damgård, J. Kilian, and L. Salvail. On the (im) possibility of basing oblivious transfer and bit commitment on weakened security assumptions. In *Eurocrypt*, pages 56–73. Springer, 1999.

[23] I. Damgård and J. Nielsen. Scalable and unconditionally secure multiparty computation. In *Crypto*, pages 572–590, 2007.

[24] I. Damgård, V. Pastro, N. Smart, and S. Zakarias. Multiparty computation from somewhat homomorphic encryption. In *Crypto*, pages 643–662, 2012.

[25] V. Dani, V. King, M. Movahedi, and J. Saia. Brief announcement: breaking the o(nm) bit barrier, secure multiparty computation with a static adversary. In *PODC*, pages 227–228, 2012.

[26] D. Dolev, C. Dwork, O. Waarts, and M. Yung. Perfectly secure message transmission. In *FOCS*, pages 36–45, 1990.

[27] S. Even, O. Goldreich, and A. Lempel. A randomized protocol for signing contracts. In *Crypto*, pages 205–210, 1983.

[28] M. Fitzi, M. Franklin, J. A. Garay, and H.V. Simhadri. Towards optimal and efficient perfectly secure message transmission. In *TCC*, pages 311–322, 2007.

[29] J. A. Garay, J. Katz, C.-Y. Koo, and R. Ostrovsky. Round complexity of authenticated broadcast with a dishonest majority. In *FOCS*, pages 658–668, 2007.

[30] M. Garey and D. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

[31] O. Goldreich, S. Micali, and A. Wigderson. How to play any mental game, or a completeness theorem for protocols with honest majority. In *STOC*, pages 218–229, 1987.

[32] O. Goldreich, S. Micali, and A. Wigderson. Proofs that Yield Nothing But Their Validity for All Languages in NP Have Zero-Knowledge Proof Systems. *J. ACM*, 38(3), pages 691–729, 1991.

[33] O. Goldreich and R. Vainish. How to solve any protocol problem - an efficiency improvement. In *Crypto*, pages 73–86, 1988.

[34] S. Goldwasser, Y. Kalai, R. Popa, V. Vaikuntanathan and N. Zeldovich. Reusable garbled circuits and succinct functional encryption. In *STOC*, pages 555–564, 2013.

[35] I. Haitner. Semi-honest to malicious oblivious transfer—the black-box way. In *TCC*, pages 412–426, 2008.

[36] D. Harnik, Y. Ishai, and E. Kushilevitz. How many oblivious transfers are needed for secure multiparty computation? In *Crypto*, pages 284–302, 2007.

[37] D. Harnik, Y. Ishai, E. Kushilevitz, and J. Nielsen. OT-Combiners via Secure Computation. In *TCC*, pages 393–411, 2008.

[38] D. Harnik, J. Kilian, M. Naor, O. Reingold, and A. Rosen. On robust combiners for oblivious transfer and other primitives. In *Eurocrypt*, pages 96–113, 2005.

[39] M. Hirt, C. Lucas, and U. Maurer. A dynamic tradeoff between active and passive corruptions in secure multi-party computation. In *Crypto (2)*, pages 203–219, 2013.

[40] Y. Huang, J. Katz, and D. Evans. Efficient secure two-party computation using symmetric cut-and-choose. In *Crypto (2)*, pages 18–35, 2013.

[41] Y. Huang, J. Katz, V. Kolesnikov, R. Kumaresan, and A. Malozemoff. Amortizing garbled circuits. In *Crypto (2)*, 2014.

[42] R. Impagliazzo and S. Rudich. Limits on the provable consequences of one-way permutations. In *STOC*, pages 44–61, 1989.

[43] Y. Ishai, J. Kilian, K. Nissim, and E. Petrank. Extending oblivious transfers efficiently. In *Crypto*, pages 145–161, 2003.

[44] Y. Ishai, E. Kushilevitz, Y. Lindell, and E. Petrank. On combining privacy with guaranteed output delivery in secure multiparty computation. In *Crypto*, pages 483–500, 2006.

[45] Y. Ishai, M. Prabhakaran, and A. Sahai. Founding cryptography on oblivious transfer - efficiently. In *Crypto*, pages 572–591, 2008.

[46] J. Kilian. Founding cryptography on oblivious transfer. In *STOC*, pages 20–31, 1988.

[47] V. Kolesnikov and T. Schneider. Improved garbled circuit: Free XOR gates and applications. In *ICALP*, pages 486–498, 2008.

[48] T. Kovári, V. Sós and P. Turán. On a problem of K. Zarankiewicz. Colloquium Mathematicae 3(1), pages 50–57, 1954.

[49] E. Kushilevitz. Privacy and communication complexity. In *FOCS*, pages 416–421, 1989.

[50] E. Larraia, E. Orsini, and N.P. Smart. Dishonest majority multi-party computation for binary circuits. In *Crypto (2)*, pages 495–512, 2014.

[51] Y. Lindell, B. Pinkas, N.P. Smart, and A. Yanai. Efficient constant round multi-party computation combining bmr and spdz. In *Crypto (2)*, pages 319–338, 2015.

[52] Y. Lindell and B. Riva. Cut-and-choose yao-based two-party computation with low cost in the online/offline and batch settings. In *Crypto*, pages 476–494, 2014.

[53] Y. Lindell. Fast cut-and-choose based protocols for malicious and covert adversaries. In *Crypto*, pages 1–17, 2013.

Lecture Notes in Computer Science, pages 1–17. Springer, August 2013.

[54] Y. Lindell and B. Pinkas. An efficient protocol for secure two-party computation in the presence of malicious adversaries. In *Eurocrypt*, pages 52–78, 2007.

[55] H. Maji, M. Prabhakaran, and M. Rosulek. A zero-one law for cryptographic complexity with respect to computational UC security. In *Crypto*, pages 595–612, 2010.

[56] R. Meier, B. Przydatek, and J. Wullschleger. Robuster combiners for oblivious transfer. In *TCC*, pages 404–418, 2007.

[57] P. Mohassel and B. Riva. Garbled circuits checking garbled circuits: More efficient and secure two-party computation. In *Crypto (2)*, pages 36–53, 2013.

[58] M. Naor and B. Pinkas. Efficient oblivious transfer protocols. In *SODA*, pages 448–457, 2001.

[59] J. Nielsen, P. Nordholt, C. Orlandi, and S. Burra. A new approach to practical active-secure two-party computation. In *Crypto*, pages 681–700, 2012.

[60] C. Peikert, V. Vaikuntanathan, and B. Waters. A framework for efficient and composable oblivious transfer. In *Crypto*, pages 554–571, 2008.

[61] M. Prabhakaran and V. Prabhakaran. On secure multiparty sampling for more than two parties. In *Information Theory Workshop (ITW)*, 2012.

[62] M. Rabin. How to exchange secrets by oblivious transfer. 1981.

[63] T. Rabin and M. Ben-Or. Verifiable secret sharing and multiparty protocols with honest majority. In *STOC*, pages 73–85, 1989.

[64] S. Wolf and J. Wullschleger. Oblivious transfer is symmetric. In *Eurocrypt*, pages 222–232, 2006.

[65] W. Jürg Wullschleger. Oblivious-Transfer Amplification. In *Eurocrypt*, pages 555–572, 2007.

[66] A. C.-C. Yao. How to generate and exchange secrets. In *FOCS*, pages 162–167, 1986.

[67] S. Zahur, M. Rosulek, and D. Evans. Two halves make a whole - reducing data transfer in garbled circuits using half gates. In *Eurocrypt*, pages 220–250, 2015.

[68] M. Zamani, M. Movahedi, and J. Saia. Millions of millionaires: Multiparty computation in large networks. In *ePrint 2014/149*.

Appendix A Bounding the number of edges in $\approx \frac{n}{2}$-unsplittable graphs.

In this section we discuss the minimum number of edges in a graph which is $(n-t)$-unsplittable for $t = \lfloor (n+1)/2 \rfloor$. We show that the minimum edge count is $n/2$ if n is even and $t = n/2$, and $(n+3)/2$ if n is odd and $t = (n+1)/2$. These bounds give the minimum number of OT channels required to obtain t-secure MPC among n parties in a network for $t = \lfloor (n+1)/2 \rfloor$. They constitute the first nontrivial cases, since no OT channels are needed in the case of an honest majority.

Theorem 20. *Let n be even and $t = n/2$. Then any $(n-t)$-unsplittable graph must contain at least t edges. This bound is tight.*

Proof. To show that the bound is tight, note that the t-claw graph (Figure 4(a)) (or any graph with a tree on $t+1$ vertices and no other edges) has t edges and contains a connected component consisting of $t+1 = n/2+1$ vertices, so it is $(n/2)$-unsplittable. We now show that every graph with fewer edges can be split.

Let G be a graph containing $t-1$ edges. Let m be the number of connected components of G, let $C_1, C_2, \ldots, C_m$ be the components in nonincreasing order of size, and let $\alpha_i = |C_i|$, so that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$ and $\sum_{i=1}^{m} \alpha_i = n$. G is $(n-t)$-splittable if and only if there is some subset $I \subset [n]$ such that $\sum_{i \in I} \alpha_i = n/2$, that is, some subset of the values α_i sum to $n/2$.

For all natural numbers x, let $a_x = |\{i : \alpha_i \geq x\}|$ denote the number of *connected components* with size at least x, and let $b_x = |\{v : v \in C_i \wedge \alpha_i < x\}|$ denote the number of *vertices* in connected components with size smaller than x. Note that a_x counts the components of certain sizes, while b_x counts the vertices contained in components of certain sizes. Since a component of size s must contain at least $s-1$ edges, for any $x > 1$ we have that $a_x \leq (t-1)/(x-1) < n/(2x-2)$. In particular, we have that $a_2 < n/2$, $a_3 < n/4$, and $a_4 < n/6$.

Since G has at most $n/2-1$ edges, at least two vertices have degree zero, so $\alpha_1 = \alpha_2 = 1$. Consequently $b_2 \geq 2$. A set of c components containing at most $t-1$ edges can include at most $c+t-1$ vertices. Consequently, for any $x > 1$, we have that

$$b_x \geq n - (a_x + t - 1) = \frac{n}{2} + 1 - a_x.$$

In particular, we have that $b_3 > \frac{n}{4} + 1$ and $b_4 > \frac{n}{3} + 1$.

Since $\alpha_1 \geq \alpha_2 \geq \alpha_3$, the third largest component has size $\alpha_3 \leq n/3$. There are only $n/2 - 1$ edges, so $\alpha_i \leq n/2$ for all i.

Consider the following greedy algorithm. Initially let $S_2 = C_2$. For i from 3 to n, if $|S_{i-1} \cup C_i| \leq n/2$ then set $S_i = S_{i-1} \cup C_i$, and otherwise set $S_i = S_{i-1}$. We show that at the end of this loop, the set S_m will always have size $n/2$.

Since $|C_m| \leq n/2$, the sum of the sizes of the other components is at least $n/2$, so if $|S_{i+1} \cup C_i| \leq n/2$ for every i considered in the loop then the loop must terminate with $|S_1| = n/2$. Otherwise there is some i such that $|S_{i-1} \cup C_i| > n/2$. Choose the last iteration i for which this is true. Since $i \geq 3$, we must have that $\alpha_i \leq \alpha_3 \leq n/3$. If $\alpha_i > 3$, then since $n/2 - |S_i| \leq \alpha_i - 1 \leq n/3 - 1$ and $b_4 > \frac{n}{3} + 1$, we reach a contradiction with the assumption that i is the last such iteration, since there are enough vertices in components $C_{i+1}, \ldots, C_m$ to make some subsequent $|S_{j-1} \cup C_j| > n/2$. Otherwise $\alpha_i \leq 3$, so since $n/2 - |S_i| \leq \alpha_i - 1 \leq 2$, so since $b_2 \geq 2$, we must have that $|S_m| = n/2$.

Consequently at the end of the loop we always have that $|S_m| = n/2$. Since S_m consists only of entire connected components, S_m consists of $n/2$ vertices with no edges to the rest of the graph, so $G \subset \Lambda^0_{n/2}$ cannot be $(n/2)$-unsplittable. □

Theorem 21. *Let n be odd and $t = (n+1)/2$. Then any $(n-t)$-unsplittable graph must contain at least $t+1$ edges. This bound is tight.*

Proof. To show that the bound is tight, note that the $(t+1)$-edge graph consisting of a cycle on $t+1$ vertices is $(n-t)$-unsplittable, since removing any vertex still leaves a connected component of size $t-1$. We now show that every graph with fewer edges can be split.

Let $G = (V, E)$ be a graph containing $|E| = t$ edges. If the maximum degree is 1, then G contains at most $n/2 < t$ edges, a contradiction. Let v be a vertex of maximal degree ≥ 2. The induced subgraph on vertex set $V \setminus \{v\}$ contains $n-1$ vertices and at most $t - 2 < (n-1)/2$ edges. By Theorem 20 it cannot be $((n-1)/2)$-unsplittable, so G cannot be $((n-1)/2)$-unsplittable either. □