# Diffie-Hellman Problems and Bilinear Maps

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**Abstract.** We investigate relations among the discrete logarithm (DL) problem, the Diffie-Hellman (DH) problem and the bilinear Diffie-Hellman (BDH) problem when we have an efficient computable non-degenerate bilinear map $e : G \times G \to H$. Under a certain assumption on the order of $G$, we show that the DH problem on $H$ implies the DH problem on $G$, and both of them are equivalent to the BDH problem when $e$ is *weak-invertible*. Moreover, we show that given the bilinear map $e$ an injective homomorphism $f : H \to G$ enables us to solve the DH problem on $G$ efficiently, which implies the non-existence a *self-bilinear* map $e : G \times G \to G$ when the DH problem on $G$ is hard. Finally we introduce a sequence of bilinear maps and its applications.

## 1 Introduction

The Weil pairing on an elliptic curve have been used to solve cryptographic problems such as the discrete logarithm (DL) problem, the (computational) Diffie-Hellman (DH) problem, the decisional Diffie-Hellman (DDH) problem [13]. After Joux proposed tripartite Diffie-Hellman protocol using the Weil paring, however, the Weil (or Tate) pairing is being used as a building block of interesting cryptographic protocols including ID-based schemes, a short signature scheme, and self-blindable credentials [9, 4, 7, 6, 18, 16].

The *bilinear* property of the pairings plays an important role on pairing-based protocols. Given two groups $G$ and $H$, a map $e : G \times G \to H$ is said to be bilinear if $e(g_1^{x_1}, g_2^{x_2}) = e(g_1, g_2)^{x_1 x_2}$ for all $x_i \in \mathbb{Z}$ and $g_i \in G$. Given a quadruple $(g, g^x, g^y, g^z)$ the bilinear Diffie-Hellman (BDH) problem asks to find $e(g, g)^{xyz}$. Though security of most paring-based protocols relies on the BDH problem, the hardness of the BDH problem or its relations with other well-known problems are not studied well.

In this paper, we investigate relations between the BDH problem and other well-known problems and how the properties of bilinear maps influence on their strength. More precisely, we showed that the DH problem on $G$ and $H$ implies the BDH problem. The DH problem on $H$ implies the DH problem on $G$ if the order of $G$ satisfies a certain condition which is believed to be true for almost all primes [11, 12].

To show the inverse direction of the above implications, we need to have an *invertible* bilinear map. Since a bilinear map is a two-variable function, invertibility can be defined several ways. A bilinear map is said to be *weak-invertible* if one can efficiently compute an

inverse image $(g_1, g_2)$ of $h$ such that $e(g_1, g_2) = h$ for any $h \in H$. Under the weak-invertible assumption of a bilinear map, we show that the DH problem on $H$, and so the DH problem on $G$ on a certain assumption, are equivalent to the BDH problem.

We can consider a stronger notion that there exists $g \in G$ such that one can efficiently compute an inverse image $g_1$ of $h$ such that $e(g_1, g) = h$ for any $h \in H$. In this case the DH problem on $G$ is efficiently solved, and so is the BDH problem. More generally, we show that given a bilinear map $e : G \times G \to H$, if there is an injective homomorphism $f : H \to G$, then the DH problem on $G$ is efficiently solved. As a corollary, we show that an efficiently computable non-degenerate bilinear map $e_s : G \times G \to G$ does not exist on a group $G$ on which the DH problem is hard. A similar result was introduced by Verheul on the XTR groups [17].

It is natural to consider $n$-multilinear maps as an extension of a bilinear map. An $n$-multilinear map is a map from $n$-tuple of a cyclic group to another cyclic group that is linear on each variable. This map can be used to design many interesting cryptographic protocols including a non-interactive multiparty Diffie-Hellman, a broadcast encryption, and a unique signature [5]. One possible approach to find multilinear maps is to have a sequence of groups and bilinear maps between each consecutive groups. However our result implies that there should not exist efficiently computable isomorphism between any of two groups in this chain. The applications of the family includes a forward-secure Diffie-Hellman as well as all applications of multilinear maps.

The rest of the paper is organized as follows: In Section 2, we introduce bilinear maps and several DH and BDH related problems. In Section 3, we recall known relations between the DH related problems and propose a useful lemma for granularity. In Section 4, we investigate the relations as invertible properties of a bilinear map vary. In Section 5 we introduce a sequence of bilinear maps and its applications. We conclude in Section 6.

## 2 Problems related to the DH and the BDH

Let $G$ and $H$ be cyclic groups of prime order $p$. We use the multiplicative group notations. We define bilinear maps and several problems related to the DH problem and the BDH problem.

### 2.1 Problems related to the DH

**Definition 1.** *For each $g \in G$ the $DL_g$ problem is defined as follows: Given $(g, g^x)$, compute $x \in \mathbb{Z}_p$. The $DL_G$ problem asks to solve the $DL_g$ problem for arbitrary $g \in G$. We call by the $DL_g$ oracle a probabilistic algorithm to give a solution of the $DL_g$ problem. By notation, we set $DL_g(g, g^x) = x$.*

**Definition 2.** *For each $g \in G$ the $DH_g$ problem is defined as follows: Given $(g, g^x, g^y)$, compute $g^{xy}$. The $DH_G$ problem asks to solve the $DH_g$ problem for arbitrary $g \in G$. We*

*call by the $DH_g$ oracle a probabilistic algorithm to give a solution of the $DH_g$ problem. By notation, we set $DH_g(g, g^x, g^y) = g^{xy}$.*

**Definition 3.** *For each $g \in G$ the $DDH_g$ problem is defined as follows: Given $(g, g^x, g^y, g^z)$, decide if $xy = z \bmod p$. The $DDH_G$ problem asks to solve the $DDH_g$ problem for arbitrary $g \in G$. We call by the $DDH_g$ oracle a probabilistic algorithm to give a decision of the $DDH_g$ problem. By notation, we set $DDH_g(g, g^x, g^y, g^z) = 1$ if $xy = z \bmod p$ and 0 otherwise.*

**Notation.** Suppose $A$ and $B$ are problems. Throughout this paper, '$A \longrightarrow B$' means that $A$ is a strong problem than $B$, that is if there is a polynomially-bounded algorithm $\mathcal{A}_A$ solving the problem $A$ then we can build another polynomially-bounded algorithm $\mathcal{A}_B$ with polynomially-bounded access to $\mathcal{A}_A$ which solves the problem $B$. '$A \longleftrightarrow B$' means '$A \longrightarrow B$' and '$B \longrightarrow A$'. It is trivial that $DL_g \longrightarrow DH_g$ and $DH_g \longrightarrow DDH_g$.

### 2.2 Problems related to the BDH

A map $e : G \times G \to H$ is said to be *bilinear* provided that $e(g_1^{x_1}, g_2^{x_2}) = e(g_1, g_2)^{x_1 x_2}$ for all $x_i \in \mathbb{Z}/p\mathbb{Z}$ and $g_i \in G$ . We denote $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$. The Weil pairing for an elliptic curve is a good example of a bilinear map from an elliptic curve to a finite field. In this paper, we assume that the bilinear map $e$ has the following properties for practical purposes:

1. Non-degenerate: There exists a $g \in G$ such that $e(g, g) \neq 1_H$.
2. Efficient computable: There is an efficient algorithm to compute $e(g_1, g_2)$ for any $g_1, g_2 \in G$.

In fact, the original Weil pairing does not satisfy the non-degeneracy, but a modified Weil pairing defined over supersingular curve has the above properties. A modified Weil paring is described in [4].

**Definition 4.** *For each $g \in G$ the $BDL_g$ problem is defined as follows: Given $(g, g^x, g^y)$, compute $t$ such that $e(g^x, g^y) = e(g, g)^t$. The $BDL_G$ problem asks to solve the $BDL_g$ problem for arbitrary $g \in G$. We call by the $BDL_g$ oracle a probabilistic algorithm to give a solution of the $BDL_g$ problem. By notation, we set $BDL_g(g, g^x, g^y) = xy$.*

**Definition 5.** *For each $g \in G$ the $BDH_g$ problem is defined as follows: Given $(g, g^x, g^y, g^z)$, compute $e(g, g)^{xyz}$. The $BDH_G$ problem asks to solve the $BDH_g$ problem for arbitrary $g \in G$. We call by the $BDH_g$ oracle a probabilistic algorithm to give a solution of the $BDH_g$ problem. By notation, we set $BDH_g(g, g^x, g^y, g^z) = e(g, g)^{xyz}$.*

**Definition 6.** *For each $g \in G$ the $DBDH_g$ problem is defined as follows: Given $(g, g^x, g^y, g^z, h^w)$ where $h = e(g, g)$, decide if $xyz = w \bmod p$. The $DBDH_G$ problem asks to solve the $DBDH_g$ problem for arbitrary $g \in G$. We call by the $DBDH_g$ oracle a probabilistic algorithm to give a solution of the $DBDH_g$ problem. By notation, we set $DBDH_g(g, g^x, g^y, g^z, h^w) = 1$ if $xyz = w \bmod p$ and 0 otherwise.*

**Remark.** It is easy to show that the $DL_g$ problem is equivalent to the $BDL_g$ problem. Clearly, $BDL_g \longrightarrow BDH_g$ and $BDH_g \longrightarrow DBDH_g$ hold.

## 3 Relations between DH-related Problems

We defined three DH-related problems. Roughly speaking, the DL problem is equivalent to the DH problem, but the DDH problem is different from either one on most cyclic groups. We briefly summarize known results on these relations [11, 12]. Moreover we will present a useful lemma on relations between problems according to granularity.

**DL versus DH.** It is trivial that the DH problem is solved using the DL oracle. For the converse, Maurer and Wolf showed the DL problem is reduced to the DH problem for some special class of groups [11, 12]. Let $p$ be an order of a group $G$. According to their result, if one can construct an elliptic curve over $\mathbb{F}_p$ such that the elliptic curve variant DL problem on the curve is feasible (such curves always exist e.g. anomalous curves and smooth curves), then the DL problem can be solved by $O(\log^3 p)$ calls $DH$ oracle.

Unfortunately it is not clear how to construct an elliptic curve over $\mathbb{F}_p$ which has a given order. There are a few classes of curves, whose order is known, such as supersingular curves and curves with complex multiplications. For convenience, we define the following condition on a prime $p$.

**Condition** $(*)$**.** There exists an elliptic curve over $\mathbb{F}_p$ on which the DL problem is solvable.

Consequently, if $G$ has an order $p$ satisfying the condition $(*)$, then the DL problem is equivalent to the DH problem.

**DH versus DDH.** Many cryptographic protocols rely on the hardness of the DDH problem for their security. However the DDH problem is not difficult for some groups, especially for supersingular elliptic curves [3]. Moreover Joux and Nguyen constructed elliptic curve groups where the DDH problem is easy while the DH problem is equivalent to the DL problem in [10].

Actually, if there is an efficient computable bilinear map $e : G \times G \to H$, then the DDH problem on $G$ is easy. For given $(g, g^x, g^y, g^z)$, we can decide whether $z = xy \mod p$ by checking the equality between $e(g, g^z)$ and $e(g^x, g^y)$. For example, the DDH problem on supersingular curves is easy since the modified Weil pairing for supersingular curves is an efficient computable bilinear map.

### Granularity of DL and DH.

**Lemma 1.** *Let $G$ be a cyclic group of prime order $p$ and $g \in G$. We have $DL_G \longleftrightarrow DL_g$ and $DH_G \longleftrightarrow DH_g$. More precisely, for any $g_1 \in G$ we have*

1. *If we have a $DL_g$ oracle with success probability $\epsilon$, we can build a $DL_{g_1}$ oracle with success probability $\epsilon^2$ by two calls of the $DL_g$ oracle.*
2. *If we have a $DH_g$ oracle with success probability $\epsilon$, we can build a $DH_{g_1}$ oracle with success probability $\epsilon^{O(\log p)}$ by $O(\log p)$ calls of the $DH_g$ oracle.*

*Proof.* Since the $DL_G$ (resp. $DH_G$) problem implies the $DL_g$ (resp. $DH_g$) problem, the lemma follows from the two assertions.

1. Suppose that we are given a pair $(g_1, g_1^x)$ for any generator $g_1$ of $G$. We can compute $t = DL_g(g, g_1)$ and $y = DL_g(g, g_1^x)$ by two calls of the $DL_g$ oracle. Then $x = yt^{-1} \bmod p$.
2. Suppose that we are given a triple $(g_1, g_1^x, g_1^y)$ for any generator $g_1$ of $G$. Let $g_1 = g^t$ for some $t \in \mathbb{Z}_p$. Each of $g^{t^{2i}} = DH_g(g, g^{t^i}, g^{t^i})$ and $g^{t^{i+1}} = DH_g(g, g^{t^i}, g^t)$ can be computed by one call of the $DH_g$ oracle. Hence $g^{t^{-1}} = g^{t^{p-2}}$ requires $O(\log p)$ calls of $DH_g$ oracle. Since $g^{t^2xy} = DH_g(g, g^{tx}, g^{ty})$ and $g_1^{xy} = DH_g(g, g^{t^{-1}}, g^{t^2xy})$, $g_1^{xy}$ can also be computed by $O(\log p)$ calls.

Consequently, we can consider the DL problem or the DH problem without fixing a generator. However, we do not know whether the $DDH_g$ oracle can be used to build a $DDH_{g_1}$ oracle for another $g_1 \in G$. Let $g_1 = g^t$ for some $t \neq 1$. The $DDHP_g(g, g_1^x, g_1^y, g_1^z)$ decides only whether $tx \cdot ty \equiv tz \mod p$, hence this information does not helpful to decide whether $xy \equiv z \mod p$ without knowing the $t$ value.

## 4 Relations between BDH-related Problems with a Bilinear Map

We investigate relations of BDH-related problems when we have a bilinear map. Moreover we consider the situation the bilinear map is strong or weak-invertible. The notions will be defined later.

### 4.1 With a Bilinear Map

Suppose we have an efficiently computable non-degenerate bilinear map $e : G \times G \to H$.

**Theorem 1.** *Let $G$ and $H$ be cyclic groups of prime order $p$. We have $DL_h \longrightarrow DL_g$ and $DH_h \dashrightarrow DH_g$.* [3] *More precisely, for any $g_1 \in G$ we have*

1. *If we have a $DL_h$ oracle with success probability $\epsilon$, we can build a $DL_g$ oracle with success probability $\epsilon$ by one call of the $DL_h$ oracle.*
2. *Assume that $p$ satisfies the condition $(*)$. If we have a $DH_h$ oracle with success probability $\epsilon$, we can build a $DH_g$ oracle with success probability $\epsilon^{O(\log^3 p)}$ by $O(\log^3 p)$ calls of the $DH_h$ oracle.*

[3] Throughout this paper, $A \dashrightarrow B$ means that the problem $A$ implies the problem $B$ if $p$ satisfies the condition $(*)$.

*Proof.* 1. Since a given pair $(g, g^x) \in G \times G$ is reduced to a pair $(h, h^x) \in H \times H$ via the bilinear map, it is trivial that the first part of the theorem follows.
2. Since $p$ satisfies the condition $(*)$, we may assume that we are able to solve the discrete logarithm on $H$ with help of $DH_h$ oracle. Suppose that we are given a triple $(g, g^x, g^y)$. Let $h = e(g, g)$, $h_1 = e(g, g^x)$ and $h_2 = e(g, g^y)$. Actually $h_1 = h^x$ and $h_2 = h^y$. By the assumption, we can compute the discrete logarithm of $h_1$ and $h_2$ as described in [11, 12] by $O(\log^3 p)$ calls of the $DH_h$ oracle. Therefore we have $g^{xy}$.

We don't know whether the converse of Theorem 1 holds in general. However we will show in Theorem 5 that the converse is true if an inverse image of the bilinear map is efficiently computable for any element of $H$.

**BDH versus DH.**

**Theorem 2.** *Let $G$ and $H$ be cyclic groups of prime order $p$. Let $h = e(g, g)$ for $g \in G$. We have $DH_g \longrightarrow BDH_g$ and $DH_h \dashrightarrow BDH_g$. More precisely, we have*

1. *If we have a $DH_g$ oracle with success probability $\epsilon$, we can build a $BDH_g$ oracle with success probability $\epsilon$ by one call of the $DH_g$ oracle.*
2. *If we have a $DH_h$ oracle with success probability $\epsilon$, we can build a $BDH_g$ oracle with success probability $\epsilon^2$ by two calls of the $DH_h$ oracle.*

*Proof.* 1. Assume that we are given $(g, g^x, g^y, g^z)$ for $g \in G$. We can compute $g^{xy} = DH_g(g, g^x, g^y)$ by one call of the $DH_g$ oracle. Hence we get $e(g, g)^{xyz} = e(g^{xy}, g^z)$.
2. Assume that we are given $(g, g^x, g^y, g^z)$ for $g \in G$. Let $h_1 = e(g, g^x)$, $h_2 = e(g, g^y)$ and $h_3 = e(g, g^z)$. We can compute $h^{xy} = DH_h(h, h_1, h_2)$ by one call of the $DH_h$ oracle. Hence we can get $e(g, g)^{xyz} = h^{xyz} = DH_h(h, h^{xy}, h_3)$ by one more call of $DH_h$ oracle.

**DBDH versus DDH.** As mentioned before, DDH problem on $G$ has polynomial time complexity since there is a bilinear map. However we don't know whether the DDH problem on $H$ is easy or not in general.

**Theorem 3.** *Let $G$ and $H$ be cyclic groups of prime order $p$. Let $h = e(g, g)$ for $g \in G$. We have $DDH_h \longrightarrow DBDH_g$. More precisely, if we have a $DDH_h$ oracle with success probability $\epsilon$, we can build a $DBDH_g$ oracle with success probability $\epsilon$ by one call of the $DDH_h$ oracle.*

*Proof.* Assume that we are given $(g, g^x, g^y, g^z, h^w)$ where $h = e(g, g)$. Let $h_1 = e(g^x, g^y)$ and $h_2 = e(g, g^z)$. We can decide whether $xyz = w \bmod p$ by one call of $DDH_h$ oracle, i.e. $DDH_h(h, h_1, h_2, h^w)$.

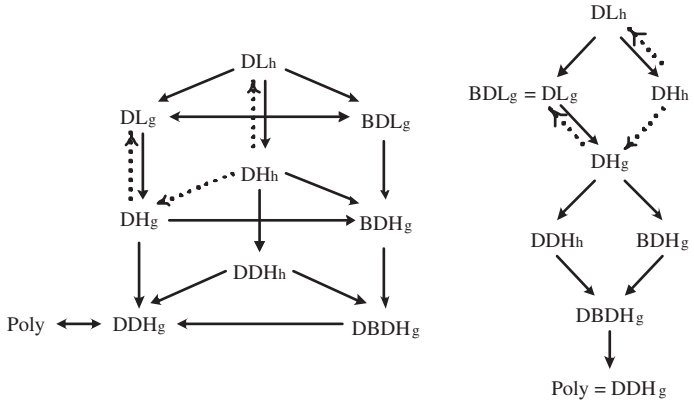


**Fig. 1.** Problems hierarchy

**Problems Hierarchy.** We can obtain the following diagram (figure 1) from the above results.

### 4.2 With a Weak-invertible Bilinear Map

**Definition 7.** *A bilinear map $e : G \times G \to H$ is said to be weak-invertible provided that an inverse image $(g_1, g_2)$ of $h$, that is $e(g_1, g_2) = h$, is efficiently computable for any $h \in H$.*

**Theorem 4.** *Suppose that $e : G \times G \to H$ is weak-invertible and $h = e(g, g)$ for $g \in G$. We have $BDH_g \longrightarrow DH_h$. More precisely, if we have a $BDH_g$ oracle with success probability $\epsilon$, we can build a $DH_h$ oracle with success probability $\epsilon^2$ by two calls of the $BDH_g$ oracle. Consequently $BDH_g \longleftrightarrow DH_h$.*

*Proof.* Since $e$ is weak-invertible, we can compute $(g_0, g_1, g_2, g_3)$ such that

$$e(g_0, g_1) = h^x, \text{ and } \quad e(g_2, g_3) = h^y.$$

Let $g_0 = g^t$ and $g_i = g_0^{a_i}$ for a positive integer $t$ and $a_i$'s. We can compute $(h^{xy})^{t^{-1}}$ by one call of $BDH_g$ oracle.

$$BDH_g(g, g_1, g_2, g_3) = e(g, g)^{t^3 a_1 a_2 a_3} = (h^{xy})^{t^{-1}}.$$

And let $(g_4, g_5)$ be an inverse image of $(h^{xy})^{t^{-1}}$, i.e. $e(g_4, g_5) = (h^{xy})^{t^{-1}}$. Finally we can compute $h^{xy}$ by one more call of $BDH_g$ oracle:

$$BDH_g(g, g_0, g_4, g_5) = e(g, g)^{t^4 a_1 a_2 a_3} = h^{xy}.$$

**Theorem 5.** *Suppose that $e : G \times G \to H$ is weak-invertible and $h = e(g, g)$ for $g \in G$. We have $DL_g \longrightarrow DL_h$ and $DH_g \longrightarrow DH_h$. More precisely,*

1. *If we have a $DL_g$ oracle with success probability $\epsilon$, we can build a $DL_h$ oracle with success probability $\epsilon^4$ by four calls of the $DL_g$ oracle. Consequently $DL_g \longleftrightarrow DL_h$.*
2. *if we have a $DH_g$ oracle with success probability $\epsilon$, we can build a $DH_h$ oracle with success probability $\epsilon^3$ by three calls of the $DH_g$ oracle.*

*Proof.* 1. For given $(h, h^x)$, we compute $g_1, g_2, g_3$ and $g_4$ such that $h = e(g_1, g_2)$ and $h^x = e(g_3, g_4)$ since $e$ is weak-invertible. Let $g_i = g^{a_i}$ for some $a_i$'s. We can compute $a_i = DL_g(g, g^i)$ by four calls of $DL_g$ oracle. Hence $x = (a_3 a_4)(a_1 a_2)^{-1}$.

2. For given $(h, h^x, h^y)$, we compute $g_i$, $(i = 1, 2, \cdots, 4)$ such that $h^x = e(g_1, g_2)$ and $h^y = e(g_3, g_4)$ since $e$ is weak-invertible. Then we can compute $g^x$ and $g^y$ by two calls of $DH_g$ oracle.

$$DH_g(g, g_1, g_2) = g^x \text{ and } DH_g(g, g_3, g_4) = g^y.$$

From the triple $(g, g^x, g^y)$, we can compute $g^{xy}$ by one more call of $DH_g$ oracle. Therefore we obtain $h^{xy} = e(g, g^{xy})$.

We can obtain the following diagram (figure 2) from the above results.

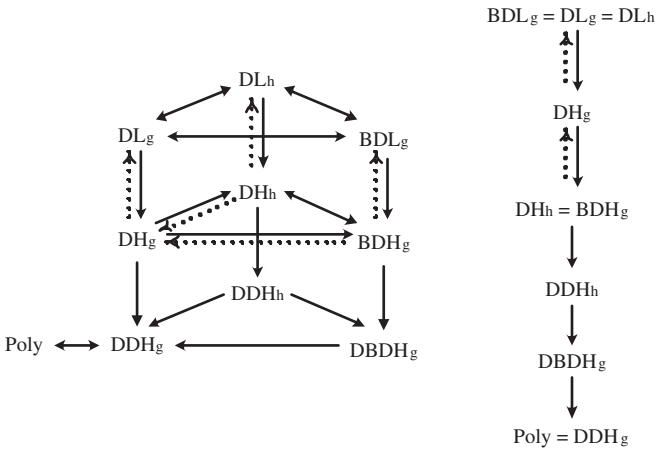


**Fig. 2.** Problems hierarchy if $e$ is weak-invertible

### 4.3 With a Strong-invertible Bilinear Map

**Definition 8.** *A bilinear map $e : G \times G \to H$ is said to be strong-invertible provided that there is an element $g \in G$ such that an inverse $g'$ of $h$ with respect to $g$, that is $e(g, g') = h$, is efficiently computable for any $h \in H$.*

If $e$ is strong-invertible with respect to $g \in G$ we can easily solve the $DH_g$ problem as follows:

**Theorem 6.** *If $e : G \times G \to H$ is strong-invertible with respect to $g \in G$ , then the $DH_g$ problem is solved using one evaluation of $e$ and one inverse operation of $e$.*

*Proof.* For given $(g, g^x, g^y)$, we first compute $e(g^x, g^y) = h_1$ and the inverse image $g_1$ of $h_1$ such that $e(g, g_1) = h_1$ by the invertibility of $e$. Since $h_1 = e(g, g)^{xy} = h^{xy}$, we have $g_1 = g^{xy}$.

If we combine the above result with Lemma 1, we can see that a strong-invertible bilinear map $e$ implies the $DH_G$ problem. Under the condition $(*)$, the $DL_G$ problem is also solved efficiently. More precisely, we have

**Corollary 1.** *If we have a strong-invertible bilinear map $e : G \times G \to H$, we can solve the $DH_G$ problem. Under the condition $(*)$, we can solve $DL_G$ problem by $O(\log^3 p)$ evaluation of $e$ and inverse operation of $e$ respectively.*

The above result can be extended to more general situation. Suppose we have an efficient computable injective homomorphism $f : H \to G$. By composition of $e$ and $f$ we can construct a self-bilinear map $e_s : G \times G \to G$. That is,

$$e_s(g_1, g_2) = f(e(g_1, g_2)).$$

$e_s$ is clearly an efficient computable non-degenerate bilinear map since $f$ is injective homomorphism.

**Lemma 2.** *Consider two functions on positive integers:*

$$f(i) = 2i + 1, \qquad g(i) = i + 1.$$

*Any $k$-bit positive integer can be generated from 1 by at most $2(k-1)$ evaluation of $f$ and $g$.*

*Proof.* Use mathematical induction on $k$. When $k = 1$, it is trivial. When $k = 2$, it is true since $10_2 = g(1)$ and $11_2 = f(1)$ where the subscript 2 denotes a binary representation. Assume the claim holds for any positive integer of at most $k - 1$ bit. Any $k$-bit integer $i$ has the form $i = 2j$ or $i = 2j + 1$ for a $(k - 1)$-bit integer $j$. Then $2j = g(f(j - 1))$ and $2j + 1 = f(j)$. Since both of $j - 1$ and $j$ are at most $(k - 1)$-bit integers, they can be computed by at most $2(k - 2)$ evaluation of $f$ and $g$. Thus $i$ can be computed by at most $2(k - 1)$ evaluation of $f$ and $g$ which proves the claim for $k$. By mathematical induction, the claim holds for any positive integer.

**Theorem 7.** *Let $G$ be a cyclic group of prime order $p$. If we have an efficiently computable non-degenerate bilinear map $e : G \times G \to G$, we can solve the $DH_G$ problem on $G$ by $O(\log p)$ evaluation of $e$.*

*Proof.* Let $g \in G$ and $e(g, g) = g^t$ for a positive integer $t$. Using Lemma 2, given a triple $(g, g^x, g^y)$, one can compute $g^{t^{-2}} = g^{t^{p-3}}$ by at most $O(\log p)$ times $e$-computations since $e(g^{t^i}, g^{t^i}) = g^{t^{2i+1}}$ and $e(g^{t^i}, g) = g^{t^{i+1}}$. Therefore we can solve the $DH_G$ as follows.

$$e(e(g^x, g^y), g^{t^{-2}}) = e(g^{txy}, g^{t^{-2}}) = g^{xy}.$$

From the above theorem, we can derive the following corollary.

**Corollary 2.** *Assume we have a non-degenerate bilinear map $e : G \times G \to H$ for two cyclic group of prime order $p$. Then there is no injective homomorphism from $H$ to $G$ if the $DH_G$ problem is hard.*

Verheul showed in [17] that if there is an injective homomorphism from the XTR subgroup to the associated supersingular curve, then the homomorphism can be utilized to make an oracle which computes the DH problem over XTR group. The proof technique is similar, but the above corollary gives the same result in more general situation.

## 5 A Sequence of Bilinear Maps

A bilinear map can be used to tripartite key agreement protocol [9]. More generally an $(n-1)$-multilinear map can be used to construct a non-interactive $n$-party key agreement protocol. This map also can be used to construct a broadcast encryption with very short broadcasts and private keys and a unique signature scheme [5].

One possible approach to construct a multilinear map is to find a sequence of cyclic groups and bilinear maps between each of consecutive two groups. Suppose that for each positive integer $n$ we have a cyclic group $G_n$ of prime order $p$ with a generator $g_n$ and an efficiently computable, non-degenerate bilinear map $e_n : G_n \times G_n \to G_{n+1}$. For each $n$-tuple $G^n$ of $G$, define $f_2 = e_1 : G_1^2 \to G_2$ and

$$f_n : G_1^n \to G_n; f_n(x_1, x_2, \cdots, x_n) = e_{n-1}(f_{n-1}(x_1, x_2, \cdots, x_{n-1}), f_{n-1}(x_n, g, \cdots, g)).$$

Then $f_n : G_1^n \to G_n$ is $n$-multilinear map for each positive $n$.

Another application of a family of bilinear maps includes a forward secure Diffie-Hellman key agreement scheme. The notion of the forward secrecy was first introduced by Anderson in 1997 to preserve the security even after the secret key has been exposed [1]. While several forward-secure signature schemes were proposed [2, 8], no forward-secure encryption scheme was announced. Note that a forward-secure encryption scheme is easily followed from a

forward-secure Diffie-Hellman key agreement scheme using ElGamal encryption technique or even using symmetric key encryption scheme.

Choose an integer $N$ whose factorization is hard and keep the factorization of $N$ to be secret. Assume we have a family of cyclic groups $G_n$ of order $N$ and efficiently computable, non-degenerate bilinear maps $e_n : G_n \times G_n \to G_{n+1}$ for each positive integer $n$. Define $g_n$ to be a generator of $G_n$ and $g_{n+1} = e_n(g_n, g_n) \in G_{n+1}$ for each positive integer $n$.

1. **Setup** Take $N$ and $G_n, e_n, g_n$ for each positive integer $n$ satisfying the above properties.
2. **Initial Keys**. A user randomly takes his initial private key $sk_1 = a$ in $\mathbb{Z}/n$. The initial public key is $pk_1 = g_1^a$.
3. **Private Key on the time** $n$. $sk_n \equiv a^{2^{n-1}} \mod N$.
4. **Public Key on the time** $n$. $pk_n = g_n^{a^{2^{n-1}}} \in G_n$
5. **Key Generation on the time** $n+1$. $sk_{n+1}$ is computed by $sk_n^2 \equiv (a^{2^{n-1}})^2 \equiv a^{2^n} \mod N$. $pk_{n+1}$ is computed by $pk_{n+1} = e_{n+1}(pk_n, pk_n) \in G_{n+1}$

Observe that the private key evolving procedure is easy, but the reverse procedure is equivalent to factoring $N$. The key evolving procedure in this scheme is very efficient since the private key evolving requires one squaring and the public key evolving requires one evaluation of $e_n$.

If we have an efficiently computable injective homomorphism $f$ from $G_j$ to $G_i$ for $i < j$, by composing it with the bilinear maps $e_n$ for $i \leq n < j$ we can construct a self-bilinear map $e : G_i \times G_i \to G_i$. For example, when $i = 1$ and $j = 4$ we can define $e : G_1 \times G_1 \to G_1$ by

$$e(x, y) = f(e_3(g_3, e_2(g_2, e_1(x, y)))) \quad \text{for some } g_i \in G_i.$$

Since a self-linear map can be used to solve efficiently the DH problem on $G_i$, our result implies that there should not exist efficiently computable isomorphism between any of two groups in this chain in order to use the family for cryptographic use.

It is not known yet whether such sequence of bilinear map influence on the security of the Diffie-Hellman problem on the base group $G_1$. But obviously the $DDH$ problem on $G_n$ can be easily solved using a bilinear map $e_n : G_n \times G_n \to G_{n+1}$.

## 6 Conclusion

We investigated relations between problems related to the DH problem when we have a bilinear map $e : G \times G \to H$. We showed the BDH problem is equivalent to the DH problem on $H$ when the bilinear map is weak-invertible. We do not know if the weak invertibility condition can be weakened. It is interesting to study how the weak-invertible property of a bilinear map influences on the security of the DH problem or the BDH problem.

We also proposed a use of a sequence of bilinear maps. This sequence has more applications than multilinear maps, but it still looks difficult to find such sequence as much as multilinear maps. We pointed out that there should not exist an efficiently computable isomorphism between any of two groups in the family. Hence the first step to construct such sequence is to find a family of groups with the same order such that there does not exist an efficiently computable isomorphism between any two of them.

## References


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